From Fermat's Last Theorem to some Generalized Fermat Equations

Nuno Freitas

Universitat de Barcelona

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Notation

- A **number field** is a finite extension $K/\mathbb{Q}$
- L is a finite extension of $\mathbb{Q}_l$
- $\mathbb{F}_{p^r}$ is the finite filed of p^r elements.
- $\mathcal{O}_k$:= Ring of integers of the filed k
- $\bar{k}$ is the algebraic closure of k
- $\bar{\mathbb{Z}}$ ring of integers of $\bar{\mathbb{Q}}$ and $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$
- **A Galois Representation** is a continous (to the Krull topology) homomorphism $\rho : G_{\mathbb{Q}} \to GL_2(L)$ or $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_{p^r})$.
- K_λ is the localization of K at the prime λ

The Modular Approach:

OBJECTIVE

Show that there are no solutions to equations with form:

(I) $x^p + 2^\alpha y^p = z^p, \quad \alpha \geq 0$

(II) $x^5 + y^5 = dz^p, \quad d = 2, 3$

The core of the approach was given by Frey, Hellegouarch, Serre, Ribet, Wiles:

(1) Construction of an elliptic Frey-Hellegouarch curve E,

(2*) Modularity results for p-adic representations $\rho_{E,p}$, attached to E

(3) Irreducibility of the mod p representations $\bar{\rho}_{E,p}$ attached to E,

(4*) Lowering the level results for representations attached to newforms $\rho_{f,p}$

(5) Contradicting the congruence $\rho_{E,p} \equiv \rho_{f,p} \pmod{\mathfrak{P}}$

(2)+(4) Serre Conjecture over $\mathbb{Q}$

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Definition

Let k be a field and $\bar{k}$ an algebraic closure of k. A **Weierstrass equation over** k is any cubic equation of the form

$$E : y^2 + a_1 y + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

where all $a_i \in k$. If $\mathrm{char}(k) \neq 2, 3$ it can be writen

$$y^2 = x^3 + Ax + B, \quad A, B \in k$$

and has discriminat $\Delta(E) = 4A^3 + 27B^2$. If $\Delta(E) \neq 0$ then E is **nonsingular** and the set

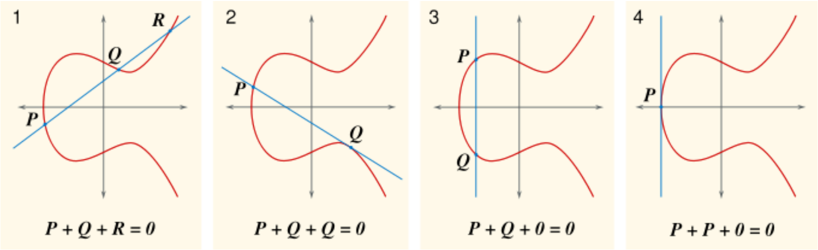
$$E = \{(x, y) \in \bar{k}^2 \text{ satisfying } E(x, y)\} \cup \{\infty\}$$

is **an elliptic curve over** k.

The Group Law

Theorem

- There is an abelian group structure on the set of points of an elliptic curves.
- (Mordell-Weil) This group is finitely generated when k is a number field.



Mod p Galois Representations attached to E

We denote by $E(\bar{\mathbb{Q}})[n]$ the points of order n.

Theorem

- $E(\bar{\mathbb{Q}})[n] \sim \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ (think over $\mathbb{C}$!)
- There is an action of $G_{\mathbb{Q}}$ on $E(\bar{\mathbb{Q}})[n]$

Let P_1, P_2 be a basis of $E(\bar{\mathbb{Q}})[n]$ and $\sigma \in G_{\mathbb{Q}}$. We can write

$$(\sigma(P_1), \sigma(P_2)) = (P_1, P_2) \begin{bmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{bmatrix}.$$

Theorem

The action of $G_{\mathbb{Q}}$ on $E(\bar{\mathbb{Q}})[n]$ defines a representation

$$\bar{\rho}_{E,n} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Z}/n\mathbb{Z}),$$

with image isomorphic $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$.

Reduction modulo p

Let $k = \mathbb{Q}$ and $E/\mathbb{Q}$ be an elliptic curve. There exists an equivalent model of E with integer coefficients such that $|\Delta(E)|$ is minimal. For such a model and a prime p we can consider the reduced curve over $\mathbb{F}_p$

$$\tilde{E} : y^2 + \tilde{a}_1 xy + \tilde{a}_3 y = x^3 + \tilde{a}_2 x^2 + \tilde{a}_4 x + \tilde{a}_6$$

and it can be seen that $\tilde{E}$ has at most one singular point.

Definition (type of reduction)

We say that E

- has **good reduction** at p if $\tilde{E}$ is an elliptic curve.
- has **bad multiplicative reduction** at p if $\tilde{E}$ admits a double point with two distinct tangents (a node)
- has **bad additive reduction** at p if $\tilde{E}$ admits a double point with only one tangent (a cusp)

The Conductor N_E

"Definition"

The **conductor** N_E of an elliptic curve E over $\mathbb{Q}$ is computed by Tate's algorithm. It is the product $\prod_p p^{f_p}$ over the primes of bad reduction of E and

$$f_p = \begin{cases} f_p = 1 \text{ if has multiplicative reduction at } p \\ f_p = 2 + \delta \geq 2 \text{ if E has additive reduction at } p, \\ f_p = 2 \text{ if E has additive reduction at } p \text{ and } p \neq 2, 3. \end{cases}$$

Definition

Let $E/\mathbb{Q}$ be an elliptic curve. We say that E is semi-stable if at every prime p the reduction of E at p is good or multiplicative.

Theorem (Mazur)

Let $p \geq 5$ be a prime and E a semi-stable elliptic curve over $\mathbb{Q}$. Then, the representation $\bar{\rho}_{E,p}$ is irreducible.

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l-adic Galois Representations attached to E

We will now attach an l-adic representation. Fix a prime l and consider the l^n-torsion sequence:

$$E[l] \xleftarrow{[l]} E[l^2] \longleftarrow E[l^3] \longleftarrow \quad \ldots$$

taking the inverse limit we have the **Tate Module at l**

$$T_l(E) = \varprojlim_n \{E[l^n]\} \cong \mathbb{Z}_l \oplus \mathbb{Z}_l.$$

From the compatibility of the action of $G_{\mathbb{Q}}$ with $[l]$ we have an action on $T_l(E)$. Since $Aut(E[l^n])$ and $GL_2(\mathbb{Z}/l^n\mathbb{Z})$ are isomorphic we also have

$$Aut(T_l(E)) \overset{\sim}{\to} GL_2(\mathbb{Z}_l),$$

hence there is a continuous homomorphism

$$\rho_{E,l} : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_l) \subset GL_2(\mathbb{Q}_l).$$

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Definition

Let p be a prime and $\mathfrak{p} \subset \bar{\mathbb{Z}}$ any maximal ideal over p. The decomposition and intertia groups at $\mathfrak{p}$ are defined by

- $D_{\mathfrak{p}} = \{\sigma \in G_{\mathbb{Q}} : \mathfrak{p}^{\sigma} = \mathfrak{p}\}$ then $\sigma \in D_{\mathfrak{p}}$ acts on $\bar{\mathbb{Z}}/\mathfrak{p} = \bar{\mathbb{F}}_p$ as $(x + \mathfrak{p})^{\sigma} = x^{\sigma} + \mathfrak{p}$
- $I_{\mathfrak{p}} = \{\sigma \in D_{\mathfrak{p}} : x^{\sigma} \equiv x \pmod{\mathfrak{p}} \text{ for all } x \in \bar{\mathbb{Z}}\}$ is the kernel of the reduction $D_{\mathfrak{p}} \to G_{\mathbb{F}_p}$.

An **absolute Frobenius element over** p is any preimage $\mathrm{Frob}_{\mathfrak{p}} \in D_{\mathfrak{p}}$ of the Frobenious automorphism in $G_{\mathbb{F}_p}$ $(x \mapsto x^p)$.
$\mathrm{Frob}_{\mathfrak{p}}$ are **dense** in $G_{\mathbb{Q}}$.

Definition

Let ρ be a Galois representation and let p be a prime. Then ρ is said to be **unramified at** p if the inertia subgroup $I_{\mathfrak{p}}$ is contained in $\mathrm{Ker}(\rho)$ for any maximal ideal $\mathfrak{p} \subset \bar{\mathbb{Z}}$ lying over p.

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Galois Representations attached to E

Let $p \nmid N_E$ be a prime of good reduction for E and define

$$a_p(E) = p + 1 - \#\tilde{E}(\mathbb{F}_p),$$

where $\#\tilde{E}(\mathbb{F}_p)$ is the number of points in the reduced curve $\tilde{E}$.

Theorem

The Galois representation $\rho_{E,l}$ is unramified at every prime $p \nmid lN_E$. For any such p let $\mathfrak{p} \subset \bar{\mathbb{Z}}$ be any maximal ideal over p. Then the characteristic equation of $\rho_{E,l}(Frob_{\mathfrak{p}})$ is

$$x^2 - a_p(E)x + p = 0.$$

The Galois representation $\rho_{E,l}$ is irreducible.

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The **modular group** $SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

and has the important **congruence subgroups**

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \text{ mod N} \right\}$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \text{ mod N} \right\}$$

where "$*$" means unspecified. Clearly, $\Gamma_1(N) \subset \Gamma_0(N)$.

Modular Forms

Let $\Gamma(N) \subset SL_2(\mathbb{Z})$ be a congruence subgroup. An holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is a **modular form of weight k with respect to $\Gamma(N)$** if

(1) For all $\tau \in \mathcal{H}$ and $\alpha \in \Gamma(N)$,

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau)$$

(2) For all $\alpha \in SL_2(\mathbb{Z})$, exists a Fourier expansion

$$(c\tau + d)^{-k} f(\frac{a\tau + b}{c\tau + d}) = \sum_{n=0}^{\infty} c_n q^{n/N}$$

where $q = e^{2\pi i \tau}$.

If in addition, $c_0 = 0$ in all the above Fourier expansions, then f is a said to be a **cusp form**. When $\alpha = \mathrm{Id}$ we denote the Fourier coefficients c_n in (2) by $a_n(f)$. Denoted by $S_k(\Gamma(N))$ the set of the cusp forms of weight k with respect to $\Gamma(N)$.

Modular Forms

- $S_k(\Gamma(N))$ is a vector space over $\mathbb{C}$ of finite dimension.
- In particular, $\mathcal{S}_2(\Gamma_0(2^t)) = \{0\}$ for $t \in \{0, 1, 2, 3, 4\}$ and $\mathcal{S}_2(\Gamma_0(32))$ has dimension 1.
- $S_k(\Gamma_1(N)) = \oplus S_k(N, \epsilon)$, where the sum is over the Dirichlet characters ϵ of modulus N.
- $f \in S_k(N, \epsilon)$ if

$$f(\frac{a\tau + b}{c\tau + d}) = \epsilon(d)(c\tau + d)^k f(\tau)$$

  for matrices in $\Gamma_0(N)$
- There are Hecke operators T_n $(n \geq 1)$ acting on $S_k(\Gamma(N))$.
- There are cuspforms that are eigenvectos for all T_n. In that case $T_n(f) = a_n(f)f$. If $f \in S_k(N, \epsilon)$ is such a form we say it is an **eigenform** of level N and character ϵ. We say f is **normalized** if $a_1(f) = 1$.

Denote by $\mathbb{Q}_f = \mathbb{Q}(\{a_p(f)\})$ the **coefficient field of** f.

Theorem

Let $f \in \mathcal{S}_k(N, \epsilon)$ be a normalized eigenform with number field $\mathbb{Q}_f$. Let l be a prime. For each maximal ideal λ of $\mathcal{O}_{\mathbb{Q}_f}$ lying over l there is an irreducible 2-dimensional Galois representation

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_{f,\lambda}).$$

This representation is unramified at every prime $p \nmid lN$. For any such p let $\mathfrak{p} \subset \overline{\mathbb{Z}}$ be any maximal ideal lying over p. Then $\rho_{f,\lambda}(Frob_{\mathfrak{p}})$ satisfies the polynomial equation

$$x^2 - a_p(f)x + \epsilon(p)p^{k-1} = 0.$$

Modular Forms

A representation of $G_{\mathbb{Q}}$ is **odd** if $\rho(c) = -1$, where c is the complex conjugation. Let χ_l be the l-adic cyclotomic character.

Definition

Let L be a finite extension of $\mathbb{Q}_l$ and consider a Galois representation $\rho : G_{\mathbb{Q}} \to GL_2(L)$. Suppose that ρ is irreducible, odd and that $\det \rho = \epsilon\chi_l^{k-1}$ where ϵ has finite image. Then ρ is **modular of level** M_f if there exists a newform $f \in \mathcal{S}_k(M_f, \epsilon)$ and a prime λ above l such that $\mathbb{Q}_{f,\lambda}$ embedds in L and such $\rho_{f,\lambda} \sim \rho$.

Modularity Theorem

Let $E/\mathbb{Q}$ be an elliptic curve. E is modular of level N_E, i.e. there is a newform $f \in \mathcal{S}_2(N_E, \epsilon = 1)$, such that $\rho_{E,l} \sim \rho_{f,l}$ for all l. In particular, $a_p(f) = a_p(E)$ for all primes $p \nmid N_E$.

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Modularity

We are also interested in modularity of mod p representations. For example, those arising from p-torsion points of elliptic curves or abelian varieties.

Definition

An irreducible representation $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$ is **modular of type** (N, k, ϵ) if there exists a newform $f \in \mathcal{S}_k(N, \epsilon)$ and a maximal ideal $\lambda \subset \mathcal{O}_{\mathbb{Q}_f}$ lying over p such that $\bar{\rho}_{f,\lambda} \sim \bar{\rho}$

Let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$ be odd and irreducible.

- Serre gives recipies compute the $N(\bar{\rho})$ (**Artin conductor**), $k(\bar{\rho})$ and $\epsilon(\bar{\rho})$.
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Modularity

- Ex. for $l = p$: if E is has multiplicative reduction at p and $p|\nu_p(\Delta)$ or E has good reduction at p then $\bar{\rho}_{E,p}$ is finite at p.
- $N(\bar{\rho})$ is divisible precisely by the primes l for which $\bar{\rho}$ is not finite and depends only on $\bar{\rho}|I_l$ for those primes.

Level Lowering Theorem

Let $p \geq 3$ be a prime. Let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_{p^r})$ be irreducible over $\bar{\mathbb{F}}_p$ and modular of type $(N, 2, 1)$. If $\bar{\rho}$ is finite at p then it is modular of type $(N(\bar{\rho}), 2, 1)$.

Serre Conjecture (Khare, Wintenberger)

Let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$ be odd and irreducible. The $\bar{\rho}$ is modular of type $(N(\bar{\rho}), k(\rho), \epsilon(\rho))$

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The Generalized Fermat Equation

We will study the solutions of the equation

$$x^p + 2^\alpha y^p + z^p = 0$$

in the following order:

- $\alpha = 0$ (Fermat's Last Theorem)
- $\alpha > 1$
- $\alpha = 1$

But first we need to introduce the Frey-Hellegouarch Curves!

Definition (ABC curve)

Let A, B, C be non-zero coprime integers such that $A + B + C = 0$ and define the elliptic curve over $\mathbb{Q}$ given by

$$E_{A,B,C}: \quad y^2 = x(x - A)(x + B)$$

that has discriminant (not always minimal) of the form $\Delta = 2^4(ABC)^2$.

Theorem

When $A \equiv -1 \bmod 4$ and $B \equiv 0 \bmod 32$, then $E_{A,B,C}$ is semi-stable and its conductor is $rad(ABC)$, the product of the primes dividing ABC.

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We need to understand the ramification of $\bar{\rho}_{E,p}$.

Theorem (Helllegouarch)

Let $C/\mathbb{Q}$ be an elliptic curve and $l \neq 2, p$. If $l \mid N_C$ is of multiplicative reduction and $p|\nu_l(\Delta(C))$ then $\bar{\rho}_{E,p}$ is unramified at l.

Néron-Ogg-Shafarevich Criterium

Let $C/\mathbb{Q}$ be an elliptic curve. C has good reduction at l if and only if $\rho_{C,p}$ is unramified at l for some prime $p \neq l$ if and only if $\rho_{C,p}$ is unramified at l for all primes $p \neq l$.

$\alpha = 0$

Suppose (a, b, c) is a **non-trivial** $(abc \neq 0)$ **primitive** (i.e. $\gcd(a, b, c) = 1$) solution of $x^p + y^p = z^p$ and let

$$A = a^p \quad B = b^p \quad C = c^p.$$

Without loss of generality we can suppose that $a \equiv -1 \pmod 4$ and b to be even.

Corollary

Let $E = E_{a^p,b^p,c^p}$. For $p \geq 5$, the representation $\bar{\rho}_{E,p}$ is unramified outside $2p$.

Proof: Let $l \neq 2, p$.

- $\Delta(E) = 2^4(ABC)^2 = 2^4(abc)^{2p}$
- If $l \nmid abc \Rightarrow l \nmid \Delta \Rightarrow E$ has good reduction at $l \Rightarrow \rho_{E,p}$ is unramified at l by N-O-S $\Rightarrow \bar{\rho}_{E,p}$ also is.
- $p \geq 5 \Rightarrow B \equiv 0 \pmod{32}$ then E is semistable. If $l \mid abc$ then by Hellgouarch theorem $\bar{\rho}_{E,p}$ is not ramified at l.

Fermat-Wiles Theorem

Let $p \geq 5$ be a prime. There are no non-trivial primitive solutions of $x^p + y^p + z^p = 0$.

Proof: Suppose that (a, b, c) is a non-trivial primitive solution. Recall $E = E_{a^p,b^p,c^p}$ is semi-stable with $\Delta = 2^4(abc)^{2p}$.

- Modularity theorem (semi-stable case) $\Rightarrow \rho_{E,p}$ is modular of level $N_E \Rightarrow \bar{\rho}_{E,p}$ is modular of level N_E.
- $\bar{\rho}_{E,p}$ is irreducible by Mazur theorem.
- $\bar{\rho}_{E,p}$ is unramified outside $2p$
- $p | \nu_p(\Delta)$ then $\bar{\rho}_{E,p}$ is finite at $p \Rightarrow N(\bar{\rho}_{E,p}) = 2$.
- We can take N_E to be $N(\bar{\rho}_{E,p})$ by the LLT.
- $S_2(\Gamma_0(2)) = \{0\} \Rightarrow \bar{\rho}_{E,p}$ is not modular, contradiction!

$\alpha = 0$

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$$a^p + 2^\alpha b^p + c^p = 0, \qquad 1 \leq \alpha \leq p-1$$

- Let (a, b, c) be non-trivial and primitive solution
- Observe that for $\alpha = 1$ there exist the solution $(-1, 1, -1)$!
- Put $A = a^p$, $B = 2^\alpha b^p$ and $C = c^p$

From Tate's algorith we have:

- $E = E_{A,B,C} : y^2 = x(x - A)(x + B)$ is semistable for $l \neq 2$.
- $N_E = 2^t rad'(ABC)$ with $t \in \{0, 1, 3, 5\}$
- $4|B$ if and only if $t \leq 3$
- $t = 5$ if and only if $ord_2(B) = 1$

Theorem

Let $p \geq 5$ be a prime and $\alpha > 1$. The equation $x^p + 2^\alpha y^p + z^p = 0$ has no non-trivial primitive solutions.

Proof: Recall that $N_E = 2^t rad'(ABC)$ and $\Delta = 2^s(abc)^{2p}$

- Modularity theorem $\Rightarrow \rho_{E,p}$ is modular of level $N_E \Rightarrow \bar{\rho}_{E,p}$ is modular of level N_E.
- Suppose $\bar{\rho}_{E,p}$ irreducible for $p \geq 5$ (Mazur do not apply!)
- $\bar{\rho}_{E,p}$ unramified outside $2p$
- $\bar{\rho}_{E,p}$ is finite at $p \Rightarrow N(\bar{\rho}_{E,p}) = 2^t$.
- We can take N_E to be $N(\bar{\rho}_{E,p})$ by LLT
- $\mathcal{S}_2(\Gamma_0(2^t)) = \{0\}$ for $t \in \{0, 1, 2, 3, 4\}$ and $\mathcal{S}_2(\Gamma_0(32))$ has dimension 1.
- $N(\bar{\rho}_{E,p}) = 2^t \Rightarrow t = 5 \Rightarrow ord_2(B) = ord_2(2^\alpha b^p) = 1$, contradiction with $\alpha > 1$ or b even

Theorem

The representation $\bar{\rho}_{E,p}$ is irreducible for $p \geq 5$.

Proof: Recall $N_E = 2^t rad'(ABC)$ with $t \in \{0, 1, 3, 5\}$

- Suppose E semistable ($t = 0, 1$). Follows from Mazur theorem.
- E not semistable $\Rightarrow$ the 2-part of N_E is $2^{2+\delta} \Rightarrow \delta = 1, 3$
- Suppose $\bar{\rho}^{ss}|I_2 = \epsilon_1 \oplus \epsilon_2$ is reducible
- $\delta = \text{cond}(\epsilon_1) + \text{cond}(\epsilon_2)$
- $\det \bar{\rho} = \bar{\chi}_p = \epsilon_1\epsilon_2$ is unramified at 2 $\Rightarrow \epsilon_2 = \epsilon_1^{-1}$
- Then $\delta = 2\,\text{cond}(\epsilon_1)$ is even, contradiction.
- Thus $\bar{\rho}|I_2$ is irreducible $\Rightarrow \bar{\rho}$ irreducible.

Observe that $E_0 = E_{(-1,1,-1)}$ by Modularity and LLT must correspondo to the eigenform in $\mathcal{S}_2(\Gamma_0(32))$. The same is true for any other $E_{(a,b,c)}$.

Proposition

If $p \equiv 1 \bmod 4$, then the image of $\bar{\rho}_{E_0,p}$ is contained in the normalizer of a Cartan split subgroup of $GL_2(\mathbb{F}_p)$.

Mazur-Momose Theorem

Let $p \geq 17$ and $C/\mathbb{Q}$ be an elliptic curve. If the image of $\bar{\rho}_{C,p}$ is contained in the normalizer of a Cartan split subgroup of $GL_2(\mathbb{F}_p)$ then C can not have multiplicative reduction at primes primes $l \neq 2$.

Theorem

Let $p \geq 17$ and $p \equiv 1 \bmod 4$. Let (a, b, c) be non-trivial primitive solution of $x^p + 2y^p + z^p = 0$. Then $(a, b, c) = (-1, 1, -1)$.

Proof:

- We can suppose that a, b, c are all odd.
- $N_E = 2^t rad'(ABC) \Rightarrow E$ has multiplicative reduction at all odd primes dividing abc.
- Since $p \equiv 1 \bmod 4$ and $\bar{\rho}_{E,p} \equiv \bar{\rho}_{E_0,p}$ by the proposition $\bar{\rho}_{E,p}$ is under Mazur-Momose hypothesis.
- Then by Mazur-Momose E has no primes of multiplicative reduction hence $abc = \pm 1$
- Thus, the only normalized solution is (-1,1,-1).

Now we proceed to the generalized equation!

Theorem (Billerey and Billerey, Dieulefait)

Let $d = 2^\alpha 3^\beta 5^\gamma$ where $\alpha \geq 2, \beta, \gamma, \geq 0$, or $d = 7, 13$. Then, for $p > 19$ the equation $x^5 + y^5 = dz^p$ has no non-trivial primitive solution.

Let γ be an integer divisible only by primes $l \not\equiv 1 \pmod 5$.

Theorem (Dieulefait, F)

For any $p > 13$ such that $p \equiv 1 \bmod 4$ and $p \equiv \pm 1 \bmod 5$, the equation $x^5 + y^5 = 2\gamma z^p$ has no non-trivial primitive solutions.

Theorem (Dieulefait, F)

For any $p > 73$ such that $p \equiv 1 \bmod 4$, the equation $x^5 + y^5 = 3\gamma z^p$ has no non-trivial primitive solutions.

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Relating two equations

Let (a, b, c) be a primitive solution to $x^5 + y^5 = d\gamma z^p$. From

Key factorization:
$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4) = (x + y)\phi(x, y)$

can be seen that

We need to prove that $\phi(x, y) = rz^p$ where $r = 1, 5$ has no non-trivial primitive solutions if $d \mid a + b$.

Observe that over $\mathbb{Q}(\sqrt{5})$

- $\phi(x, y) = \phi_1(x, y)\phi_2(x, y)$, where
- $\phi_1(x, y) = x^2 + \omega xy + y^2$ and $\phi_2(x, y) = x^2 + \bar{\omega}xy + y^2$, with
- $\omega = \frac{-1+\sqrt{5}}{2}$, $\bar{\omega} = \frac{-1-\sqrt{5}}{2}$

The Frey $\mathbb{Q}$-curve

Let (a, b, c) be a primitive solution of $\phi(x, y) = rz^p$.

Definition (Frey-curve)

Consider over $\mathbb{Q}(\sqrt{5})$ the curve given by

$$E_{(a,b)} : y^2 = x^3 + 2(a+b)x^2 - \bar{\omega}\phi_1(a, b)x,$$

with $\Delta(E) = 2^6\bar{\omega}\phi\phi_1$, where

- There are Galois representations $\rho_{E,l}$ and $\bar{\rho}_{E,l}$ of $G_{\mathbb{Q}}(\sqrt{5})$
- We need to extend them to $G_{\mathbb{Q}}$ and compute $(N(\bar{\rho}), k(\bar{\rho}), \epsilon(\bar{\rho}))$ to apply Serre conjecture

From Serre conjecture there is a newform f of type $(M, 2, \bar{\epsilon})$ with $M = 1600$, 800, 400 or 100 and a prime $\mathfrak{P}$ in $\mathbb{Q}_f$ above p such that $\bar{\rho} \equiv \bar{\rho}_{f,\mathfrak{P}} \pmod{\mathfrak{P}}$

Observe that $\mathbb{Q}(i) = \mathbb{Q}(\bar{\epsilon}) \subseteq \mathbb{Q}_f$ and define the sets:

S1: Newforms with CM (Complex Multiplication),

S2: Newforms without CM and field of coefficients strictly containing $\mathbb{Q}(i)$,

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