An Approach to Mathematical Finance

David Ruiz

UiO : **Faculty of Mathematics and Natural Sciences**
University of Oslo

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Financial Market

Definition (Financial Market)

A Financial Market is a market in which people and companies can trade financial securities, commodities, stocks or other equities or assets.

Example

Financial securities include *stocks* or *bonds*. A market of commodities include metals, oil, agricultural goods, ...

Stochastic process

Definition (Random variable)

Given a probability space $(\Omega, \mathcal{F}, P)$, *with* Ω *a sample space,* $\mathcal{F}$ *its* σ*-algebra of events and* P *a probability. A random variable is a measurable function or map* $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition (Stochastic process)

A stochastic process is a function $X : [0, T] \times \Omega \to \mathbb{R}$ *such that, for all fixed* $t \in [0, T]$, $X(t, \cdot)$ *is a random variable and for all fixed* $\omega \in \Omega$, $X(\cdot, \omega)$ *is a real-valued ordinary function.*

So, a stochastic process is simply a family (countable or uncountable) of random variables.

Example

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Also, fixed t, let's say $t =$"tomorrow", we can have, $S_t \sim N(12, 2.5)$.
In reality, $S_t(\omega)$ is not *deterministic*, but *random*.

Martingale Process

We say that a stochastic process $M_t(\omega)$ is a P-martingale if (integrable + adapted)

$$E(M_t|\mathcal{F}_s) = M_s \text{ for all } t \geqslant s.$$

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$$E(M_t|\mathcal{F}_s) \leqslant M_s \quad \text{Supermartingale} \rightarrow \text{Game favouring counterpart.}$$

The Market Model

We consider $n+1$, financial securities, let's say stocks, $S_t^0(\omega), S_t^1(\omega), \ldots, S_t^n(\omega)$, where the first one is risk-less and the other n are risky.

The Market Model

The risk-less financial security, measured in domestic currency, is driven by the following stochastic differential equation

$$\begin{cases} dS_t^0(\omega) = S_t^0(\omega) r(t,\omega) dt, \\ S_0^0(\omega) = 1. \end{cases}$$

Here, r is called the interest rate and it does not need to be deterministic. This could, for instance, be a *bank account* or *bond*, although bonds may *default*.

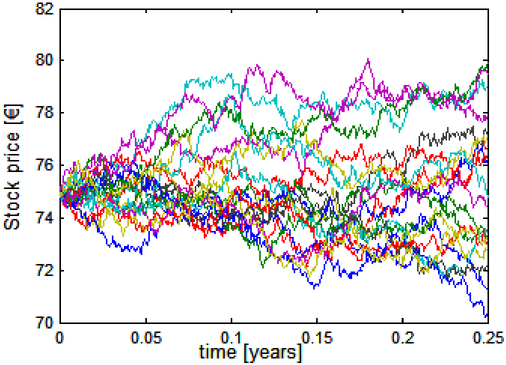
The Market Model

The other n stocks are driven by the following SDE

$$\begin{cases} dS_t^i(\omega) = S_t^i(\omega)(b^i(t,\omega)dt + \sum_{k=1}^n \sigma^{i,k}(t,\omega)dW_t^i(\omega)), \\ S_0^i(\omega) > 0, \end{cases}$$

where S_t^i is the i-th security asset.
W_t is a (vector) stochastic process with infinite variation. dW_t is a "kind of differential", a noise.
b is called the *drift* (a vector) and σ is the *volatility* (a matrix).

The Market Model



Example of modelling stock prices using brownian motion.

Portfolio

Definition (Portfolio)

A portfolio or strategy is a vector stochastic process $\theta = (\theta_t^0(\omega), \theta_t^1(\omega), \ldots, \theta_t^n(\omega))$, *where each* $\theta_t^i(\omega)$ *denotes the number of units invested in the stock* i *at time* $t \in [0, T]$, *and* $i = 0, \ldots, n$.

Definition

The value of a portfolio denoted by $V_t^\theta(\omega)$ *is given by the discrete scalar product,*

$$V_t^\theta(\omega) = \sum_{i=0}^{n} \theta_t^i(\omega) S_t^i(\omega) = \theta \cdot S$$

Portfolio

Definition (Admissibility)

A portfolio θ is said to be admissible if its value is almost surely non-negative, i.e.: $V_t^\theta(\omega) \geqslant 0$, P-a.s.

Definition (Self-financing portfolio)

We say that a portfolio θ is self-financing if $dV_t^\theta = \theta_t^0 dS_t^0 + \sum_{i=1}^n \theta_t^i dS_t^i$.

Hence also,

$$V_t = V_0 + \int_0^t \theta_s^0 dS_s^0 + \sum_{i=1}^n \int_0^t \theta_s^i dS_s^i.$$

Arbitrage opportunity

Definition (Arbitrage opportunity)

A portfolio or strategy θ is said to be an arbitrage opportunity if $V_0^\theta = 0$ and there exists a time $t \in [0, T]$ such that $V_t^\theta > 0$, with strictly positive probability. That is, $P(\omega : V_t^\theta(\omega) > 0) > 0$.

Derivative

A *derivative* or *option* is a contract where the *seller* of this contract gets an amount of money from the *buyer* of this contract.

Definition (Contingent Claim)

A contingent claim is an $\mathcal{F}_T$-measurable, and positive random variable.

Example: A European call option.

Fair price

Imagine we have an option with a final payoff $h(\omega)$ at maturity time T. How much should the buyer of this option pay now?

Definition (Replicating portfolio)

Consider a contingent claim or payoff h. A replicating or hedging portfolio for h is a self-financing portfolio θ such that its final value is h, i.e.: $V_T^\theta(\omega) = h(\omega)$, P-a.s.

Example: A European call option. The buyer of this option gets $h = (S(T) - K)_+$.

Fair price

The price of a contract now $(t = 0)$ with payoff h at time T is the initial value of a portfolio such that it replicates h.

$$\text{Fair price now} = V_0^\theta \quad \text{with} \quad V_T^\theta = h.$$

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Definition (Complete Market)

A financial market is said to be complete, if any contingent claim is replicable.

Arbitrage opportunity example

A market bond/stock		
Asset/Time	Today t=0	Tomorrow t=1
Bond B_t	0.95 $	1 $
Stock S_t	1 $	2 $ $\omega = \omega_1$ 1/2 $ $\omega = \omega_2$

What is the fair price of European call $C_1 = (S_1 - K)_+$ with strike $K = 1$?

Arbitrage opportunity example

Example: A car in USA is cheaper than in Canada.

Americans: Buy car in USA, drive to Canada, sell more expensive and sell CAD they get for the sale.

Canadians: Buy USD to buy a car in USA and return home.

The exploit of this arbitrage makes supply of CAD and demand for USD increase. This implies USD increase in price and CAD decrease, this would make the price of American cars more expensive until the arbitrage situation disappears.

A Pricing Formula

$$p_t = S_t^0 E_{P^*}\left[\frac{h}{S_T^0}|\mathcal{F}_t\right],$$

where P^* is a different probability, equivalent to P (the risk neutral probability).

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The factor we use to discount is called *numéraire* or *benchmark*.

There are other formulas changing measure and numéraire.

Pricing Theory

(A)

(B)

(C)

(A) Robert Merton (born 31 July 1944, aged 67): American economist.
(C) Myron Scholes (born July 1, 1941, aged 70): Canadian-born American financial economist.
(B) Fischer Black (January 11, 1938 – August 30, 1995): American economist.

Risk Minimization

Let h be a claim which is not attainable. This means that an amissible portfolio θ whose final value is:

$$V_T^{\theta}(\omega) = h(\omega)$$

can not be self-financing. So, it has a cost

Risk Minimization

Not all markets are complete! i.e.: Not all claims are replicable or attainable.

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The cost of a non-self-financing strategy at time $t \in [0, T]$ is

$$C_t^{\theta}(\omega) = V_t^{\theta}(\omega) - V_0^{\theta}(\omega) - \int_0^t \theta_s(\omega) \cdot dS_s(\omega)$$

which represents the part of the value that has not been gained from trading.

Risk Minimization

Definition (Risk)

Given a non-self-financing strategy θ, and C its cost process, we define the risk of this strategy at time $t \in [0, T]$ as

$$R_t = E\left[\left(\overline{C}_T - \overline{C}_t\right)^2 |\mathcal{F}_t\right],$$

where $\overline{C} = \frac{C}{S^0}$. This is, the mean square of its remaining cost.

Risk Minimization

Substituting we have that the risk of a non self-financing strategy is:

$$R_t = E\left[\left(\overline{h} - \overline{V}_t^{\theta} - \int_t^T \theta_s d\overline{S}_s\right)^2 |\mathcal{F}_t\right]$$

which can be seen as a way to measure how well the current value of the portfolio plus future trading can approximate the non-attainable claim.

References

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Thank you!