A Parameterization Method for Computing Normally Hyperbolic Invariant Tori

Some Numerical Examples

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Definitions and introducing the problem

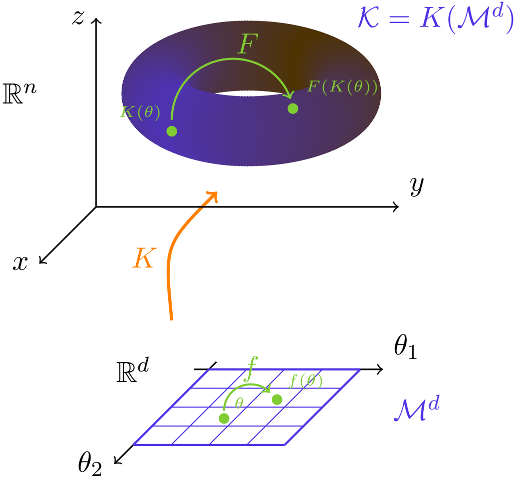
The system model

- Consider a map in $\mathbb{R}^n$: $F : \mathbb{R}^n \to \mathbb{R}^n$ (a discrete dynamical system).

- Consider a d-manifold $\mathcal{M}^d = \left\{ \mathbb{T}^d, \mathbb{S}^d, \dots \right\} \subset \mathbb{R}^n$.
  A ***parameterization*** of it will be an immersion $K : \mathcal{M}^d \to \mathbb{R}^n$ $(DK$ has maximum rank $d)$, $d < n$, $\mathcal{K} = K(\mathcal{M}^d)$.

- Let $f : \mathcal{M}^d \to \mathcal{M}^d$ be the dynamics of F restricted over the manifold $\mathcal{M}^d$.

The system model



The Invariance equation

Definition

We say that the manifold parameterized by K, $\mathcal{K}$, is invariant under F with internal dynamics f if K and f meets the invariance equation:

$$F \circ K = K \circ f$$

ie: if for each $\theta \in \mathcal{M}^d$ we have $F(K(\theta)) = K(f(\theta))$.

We want to find these functions K and f.

Vector Bundles, hyperbolicity and Normal Hyperbolicity

Consider $M \subset \mathbb{R}^n$ a manifold.

For one point of the manifold $M \rightsquigarrow$ **vector spaces**:

- T_pM, tangent space to M at p.
  Particulary, $T_p\mathbb{R}^n = \mathbb{R}^n$ the tangent space to $\mathbb{R}^n$ at p.
- $N_p = \{v \in \mathbb{R}^n | v \perp T_pM\}$, normal space to M at p.
- $T_p\mathbb{R}^n = T_pM + N_p$, as a vectorial spaces sum.

Vector Bundles, hyperbolicity and Normal Hyperbolicity

For all points (together) of the manifold $M \rightsquigarrow$ **vector bundles**: a manifold such which have a linear vector space associated to every point of it:

- $TM = \{(p,v) \in M \times \mathbb{R}^n | v \in T_pM\}$, tangent bundle of M.
- $N = \{(p,v) \in M \times \mathbb{R}^n | v \perp T_pM\}$, normal bundle of M.
- $T\mathbb{R}^n_{|M} = TM \oplus N$, as a Whitney sum of the vector bundles.

For each $p \in M$ we have $T_pM \cong \mathbb{R}^d$ and $N_p \cong \mathbb{R}^{n-d}$, called the *fibers* of the vector space.

Suppose we have found $\mathcal{K} = K(\mathcal{M}^d)$ and f.

- DK generates the tangent space to each point of the invariant manifold, $T_{K(\theta)}\mathcal{K}$:

$$(DK)_\theta : T_\theta \mathcal{M}^d \to T_{K(\theta)}\mathcal{K}$$

  for each $\theta \in \mathcal{M}^d$, $DK(\theta)$ is represented as a $n \times d$ matrix, a fiber.
  Considering all $\theta \in \mathcal{M}^d$, we have a parameterization of the tangent bundle, $T\mathcal{K}$.

- if $N(\theta)$ is a $n \times n - d$ matrix composed by $n - d$ vectors linearly independent to the vectors of $DK(\theta)$, then it generates the normal space $\mathcal{N}_{K(\theta)}$, complementary to $T_{K(\theta)}\mathcal{K}$.
  Considering all $\theta \in \mathcal{M}^d$, we have a parameterization of the normal bundle, $\mathcal{N}(\mathcal{K})$.

So, we could write:

$T_{K(\theta)}\mathbb{R}^n = T_{K(\theta)}\mathcal{K} + \mathcal{N}_{K(\theta)} \cong \mathbb{R}^n$ as a sums of vector spaces.

$T\mathbb{R}^n_{|\mathcal{K}} = T\mathcal{K} \oplus \mathcal{N}(\mathcal{K})$ as a sums of vector bundles.

We could define $P(\theta) := (DK(\theta)|N(\theta))_{n\times n}$, a vector bundle which generates the total space. So, it is an adapted frame of if

$$P : \mathcal{M}^d \to T_{K(\theta)}\mathcal{K} + \mathcal{N}_{K(\theta)}$$

We could also define the matrix map $\Lambda : \mathcal{M}^d \to M_{n\times n}$ as the dynamics over the tangent and normal bundles, the linearized dynamics on this frame.

Then, Λ and P must satisfy the invariance of the splitting on the bundles:

$$P(f(\theta))^{-1} DF(K(\theta))P(\theta) = \Lambda(\theta)$$

Remark

The dynamics of the bundles on the model manifold will be of the form

$$\Lambda(\theta) = \begin{pmatrix} \Lambda^t(\theta) & B(\theta) \\ O & \Lambda^n(\theta) \end{pmatrix}$$

where Λ^t is the linearized dynamics on the tangent space (a $d \times d$ matrix), Λ^n is the linearized dynamics on the normal space (a $n - d \times n - d$ matrix) and $B(\theta)$ is a $d \times n - d$ matrix.

As

$$DF(K(\theta))\,(DK(\theta)|N(\theta)) = (DK(f(\theta))|N(f(\theta)))\begin{pmatrix} \Lambda^t(\theta) & B(\theta) \\ O & \Lambda_n(\theta) \end{pmatrix}:$$

- If $B(\theta) = 0$, then the normal bundle is invariant: the invariance equation is satisfied on the normal subspace

$$DF(K(\theta))N(\theta) = N(f(\theta))\Lambda^n(\theta)$$

- If $\Lambda^t(\theta) = Df(\theta)$, then the tangent bundle is invariant: the invariance equation is satisfied on the tangent subspace

$$DF(K(\theta))DK(\theta) = DK(f(\theta))Df(\theta)$$

  This is always true, as this equation is just the derivative of invariance equation.

So, we will consider $B(\theta) = 0$ and $\Lambda^t(\theta) = Df(\theta)$ to have last two conditions true and have both bundles invariant. In this way, the invariant manifold will be normally hyperbolic.

The method

- The algorithm is inspired in the **parameterization method** *(Cabré,Fontich, de la Llave)* for finding a parameterization of the invariant manifold and a dynamics on it.

- The framework leads to **solving invariance equations**, for which one uses **a Newton method** adapted to the dynamics and the geometry of the (invariant) manifold, normally hyperbolic invariant tori.

Summary of the problem

If we want to find a normally hyperbolic invariant manifold, we are looking for K, f, P and Λ such that:

- $F(K(\theta) - K(f(\theta)) = 0$
- $DF(K(\theta)P(\theta) - P(f(\theta))\Lambda(\theta) = 0$

$$DF(K(\theta)N(\theta) - N(f(\theta))\Lambda^n(\theta) = 0$$

But as tangent bundle and its dynamics are well defined directly using derivatives of known values K and f, so we only have to solve the second equation for the normal part.

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A newton method to compute K, f, P, Λ

Given an approximate normally hyperbolic invariant torus (K_0, f_0) and its bundles (P_0, Λ_0) with error:

- $F(K_0(\theta) - K_0(f_0(\theta)) = R(\theta)_{n\times 1}$
- $DF(K_0(\theta))P_0(\theta) - P_0(f_0(\theta))\Lambda_0(\theta) = S(\theta)_{n\times n}$

We look for (H, h, Q, Δ) satisfying

- $-\tilde{R}(\theta) = \Lambda(\theta)H(\theta) - \underbrace{\begin{pmatrix} h(\theta) \\ 0 \end{pmatrix}}_{n\times d} - H(f(\theta))$
- $-\tilde{S^n}(\theta) = \Lambda(\theta)Q(\theta) - Q(f(\theta))\Lambda^n(\theta) - \Delta(\theta)$

And obtain the improved torus:

$$K(\theta) = K_0(\theta) + P_0(\theta)H(\theta), \qquad f(\theta) = f_0(\theta) + h(\theta),$$
$$N(\theta) = N_0(\theta) + P_0(\theta)Q(\theta), \qquad \Lambda^n(\theta) = \Lambda_0^n(\theta) + \Delta(\theta)$$

Consider $K(\theta) = K_0(\theta) + P_0(\theta)H(\theta)$ and $f(\theta) = f_0(\theta) + h(\theta)$ as new approximations of K and f.

We want to solve $F(K(\theta) - K(f(\theta)) = 0$, which in terms of the new approximation means:

$$0 = F(K_0(\theta) + P_0(\theta)H(\theta)) \\ - (K_0(f_0(\theta) + h(\theta)) + P_0(f_0(\theta) + h(\theta))H(f_0(\theta) + h(\theta)))$$

Doing computations, and neglecting quadratically small terms, it becomes:

$$-\tilde{R}(\theta) = \Lambda(\theta)H(\theta) - \begin{pmatrix} h(\theta) \\ 0 \end{pmatrix} - H(f(\theta))$$

where $\tilde{R}(\theta)$ is the projection over the invariant subspaces $\tilde{R}(\theta) := P_0^{-1}(f_0(\theta))R(\theta)$.

As the dynamics Λ_0 splits into tangent and normal part

$$\Lambda_0(\theta) = \begin{pmatrix} \Lambda_0^t(\theta) & 0 \\ 0 & \Lambda_0^n(\theta) \end{pmatrix} = \begin{pmatrix} Df_0(\theta) & 0 \\ 0 & \Lambda_0^n(\theta) \end{pmatrix}$$

last equation also splits into tangent and normal part:

$$\begin{aligned} -\tilde{R}^t(\theta) &= Df_0(\theta)H^t(\theta) - h(\theta) - H^t(f_0(\theta)) \\ -\tilde{R}^n(\theta) &= \Lambda_0^n(\theta)H^n(\theta) - H^n(f_0(\theta)) \end{aligned}$$

and can be solved separately.

Normal component

Also, as we suppose that the invariant manifold is hyperbolic, the linearized normal dynamics is expressed as

$$\Lambda_0^n(\theta) = \begin{pmatrix} \Lambda_0^u(\theta) & 0 \\ 0 & \Lambda_0^s(\theta) \end{pmatrix}$$

and the normal part equation also splits into stable and unstable components :

$$\begin{aligned} -\tilde{R}^s(\theta) &= \Lambda_0^s(\theta)H^s(\theta) - H^s(f_0(\theta)) \\ -\tilde{R}^u(\theta) &= \Lambda_0^u(\theta)H^u(\theta) - H^u(f_0(\theta)) \end{aligned}$$

and can be solved separately.

Normal component

We could solve both equation by simple iteration using the contracting principle, which will converge to the wanted solutions H^s, H^u.

⇝ STABLE COMPONENT : For H^s the equation is already a contraction (linearized stable dynamics Λ_0^s contracting). Doing some arrangements on it we obtain the iterating equation:

$$-\tilde{R}^s(\theta) = \Lambda_0^s(\theta)H^s(\theta) - H^s(f_0(\theta)) \dashrightarrow H^s_{N+1}(\theta) = \Lambda_0^s(f_0^{-1}(\theta))H^s_N(f_0^{-1}(\theta)) + \tilde{R}^s(f_0^{-1}(\theta))$$

⇝ UNSTABLE COMPONENT : For H^u we have the equation as an expansion (linearized unstable dynamics Λ_0^u expanding). Doing arrangements to get Λ_0^u inverted and so the equation as a contraction, the iterating equation is:

$$-\tilde{R}^u(\theta) = \Lambda_0^u(\theta)H^u(\theta) - H^u(f_0(\theta)) \dashrightarrow H^u_{N+1}(\theta) = (\Lambda_0^u(\theta))^{-1}\left[H^u_N(f_0(\theta)) - \tilde{R}^u(\theta)\right]$$

Tangent component

To solve the tangent part

$$-\tilde{R}^t(\theta) = Df_0(\theta)H^t(\theta) - h(\theta) - H^t(f_0(\theta))$$

we have to solve an overdetermined system: have one equation with two unknowns (H^t and h).

So, we have to chose some condition over the system to be able to solve it.

- Election: $H^t(\theta) = 0$, which means we don't modify the tangent part.

Doing this election, we obtain the uniqueness of the solution of the equation, which will be:

$$h(\theta) = \tilde{R}^t(\theta)$$

Solution of $F(K(\theta) - K(f(\theta)) = 0$

The improved torus and it dynamics becomes of the form

$$\begin{aligned} \hat{K}(\theta) &= K_0(\theta) + N_0(\theta)H_N^n(\theta) \\ \hat{f}(\theta) &= f_0(\theta) + \tilde{R}^t(\theta) \end{aligned}$$

with a new error $\hat{R}(\theta) = F(\hat{K}(\theta) - \hat{K}(\hat{f}(\theta))$ quadratically small than $R(\theta)$.

Suppose $\hat{K}$ and $\hat{f}$ had already been computed. The error on the bundles changes to:

$$DF(\hat{K}(\theta))P_0(\theta) - P_0(\hat{f}(\theta))\Lambda_0(\theta) = \hat{S}(\theta)_{n\times n}$$

and taking into account only the normal part of it, we consider:

$$DF(\hat{K}(\theta))N_0(\theta) - N_0(\hat{f}(\theta))\Lambda_0^n(\theta) = \hat{S}^n(\theta)_{n\times n-d}$$

Consider $N(\theta) = N_0(\theta) + P_0(\theta)Q(\theta)$ and $\Lambda^n(\theta) = \Lambda_0^n(\theta) + \Delta(\theta)$ as new approximations of bundles and bundles dynamics.
We want to solve $DF(K(\theta)N(\theta) - N(f(\theta))\Lambda^n(\theta) = 0$, which in terms of the new approximation means:

$$\begin{aligned} 0 = {} & DF(\hat{K}(\theta))\left(N_0(\theta) + N_0(\theta)Q(\theta)\right) \\ & - \left(N_0(\hat{f}(\theta)) + N_0(\hat{f}(\theta))Q(\hat{f}(\theta))\right) - \left(\Lambda_0^n(\theta) + \Delta^n(\theta)\right) \end{aligned}$$

Doing computations, and neglecting quadratically small terms, it becomes

$$-\tilde{S}^n(\theta) = \Lambda_0(\theta)Q(\theta) - Q(\hat{f}(\theta))\Lambda_0^n(\hat{f}(\theta)) - \begin{pmatrix} 0_{d\times n-d} \\ \Delta^n(\theta)_{n-d\times n-d} \end{pmatrix}$$

where $\tilde{S}^n(\theta)$ is the projection over the invariant subspaces, $\tilde{S}^n(\theta) := P_0^{-1}(\hat{f}(\theta))\hat{S}^n(\theta)$.

Matrix notation

For Q and S we use the notation:

$$M(\theta) = \begin{pmatrix} M^{tt}(\theta) & M^{ts}(\theta) & M^{tu}(\theta) \\ M^{st}(\theta) & M^{ss}(\theta) & M^{su}(\theta) \\ M^{ut}(\theta) & M^{us}(\theta) & M^{uu}(\theta) \end{pmatrix}_{n\times n}$$

were t, s, u means *tangent*, *stable* and *unstable* direction projections respectively.

In our case we do not modify the tangent space, so we only need lasts $n-d$ columns corresponding to the normal space:

$$M(\theta) = \begin{pmatrix} M^{ts}(\theta) & M^{tu}(\theta) \\ M^{ss}(\theta) & M^{su}(\theta) \\ M^{us}(\theta) & M^{uu}(\theta) \end{pmatrix}_{n\times n-d}$$

Using this notation, we could split the last matrix equation into 6 equations as:

$$\begin{aligned}
-\tilde{S}^{ts}(\theta) &= \Lambda_0^t(\theta)Q^{ts}(\theta) - Q^{ts}(\hat{f}(\theta))\Lambda_0^s(\hat{f}(\theta)) \\
-\tilde{S}^{ss}(\theta) &= \Lambda_0^s(\theta)Q^{ss}(\theta) - Q^{ss}(\hat{f}(\theta))\Lambda_0^s(\hat{f}(\theta)) - \Delta^s(\theta) \\
-\tilde{S}^{us}(\theta) &= \Lambda_0^u(\theta)Q^{us}(\theta) - Q^{us}(\hat{f}(\theta))\Lambda_0^s(\hat{f}(\theta)) \\
-\tilde{S}^{tu}(\theta) &= \Lambda_0^t(\theta)Q^{tu}(\theta) - Q^{tu}(\hat{f}(\theta))\Lambda_0^u(\hat{f}(\theta)) \\
-\tilde{S}^{su}(\theta) &= \Lambda_0^s(\theta)Q^{su}(\theta) - Q^{su}(\hat{f}(\theta))\Lambda_0^u(\hat{f}(\theta)) \\
-\tilde{S}^{uu}(\theta) &= \Lambda_0^u(\theta)Q^{uu}(\theta) - Q^{uu}(\hat{f}(\theta))\Lambda_0^u(\hat{f}(\theta)) - \Delta^u(\theta)
\end{aligned}$$

Projection Method

Remark

To simplify the algorithm we use the projection method: make projections into the normal direction, making zero the components projected to itself: $Q^{ss} = Q^{uu} = 0$.

Doing it, we obtain directly the correction of the linearized normal dynamics $\Lambda^n(\theta)$:

$$\tilde{S}^{ss}(\theta) = \Delta^s(\theta)$$

$$\tilde{S}^{uu}(\theta) = \Delta^u(\theta)$$

The other 4 equations can be solved iterating by the contraction principle: all equations are contractions or expansions by NHIM definition (normal dynamics always dominates the character of $\Lambda^t(\theta)$).

Doing arrangements to obtain $Q^{**}(\theta)$, $* = \{t, s, u\}$, conveniently isolated, these 4 iterating equations become:

$$\begin{aligned}
Q^{ts}_{N+1}(\theta) &= \left(Q^{ts}_N(\hat{f}(\theta))\Lambda^s_0(\theta) - \tilde{S}^{ts}(\theta)\right)(\Lambda^t_0)^{-1}(\theta) \\
Q^{us}_{N+1}(\theta) &= \left(Q^{us}_N(\hat{f}(\theta))\Lambda^t_0(\theta) - \tilde{S}^{us}(\theta)\right)(\Lambda^u_0)^{-1}(\theta) \\
Q^{tu}_{N+1}(\theta) &= \left(\Lambda^t_0(\hat{f}^{-1}(\theta))Q^{tu}_N(\hat{f}^{-1}(\theta)) + \tilde{S}^{tu}(\hat{f}^{-1}(\theta))\right)(\Lambda^u_0)^{-1}(\hat{f}^{-1}(\theta)) \\
Q^{su}_{N+1}(\theta) &= \left(\Lambda^s_0(\hat{f}^{-1}(\theta))Q^{su}_N(\hat{f}^{-1}(\theta)) + \tilde{S}^{su}(\hat{f}^{-1}(\theta))\right)(\Lambda^u_0)^{-1}(\hat{f}^{-1}(\theta))
\end{aligned}$$

and they converge to the solution when $N \to \infty$.

Solution of $DF(K(\theta)N(\theta) - N(f(\theta))\Lambda^n(\theta) = 0$

The improved bundles and it dynamics becomes of the form

- $\hat{P}(\theta) = (\hat{T}(\theta)|\hat{N}^s(\theta)|\hat{N}^u(\theta))$:

$$\begin{aligned} \hat{N}^s(\theta) &= N_0^s(\theta) + DK_0(\theta)Q^{ts}(\theta) + N_0^u(\theta)Q^{us}(\theta) \\ \hat{N}^u(\theta) &= N_0^u(\theta) + DK_0(\theta)Q^{tu}(\theta) + N_0^s(\theta)Q^{su}(\theta) \\ \hat{T}(\theta) &= D\hat{K}(\theta) \end{aligned}$$

- $\hat{\Lambda}(\theta) = \begin{pmatrix} \hat{\Lambda}^t(\theta) & 0 & 0 \\ 0 & \hat{\Lambda}^s(\theta) & 0 \\ 0 & 0 & \hat{\Lambda}^u(\theta) \end{pmatrix}$: $\begin{aligned} \hat{\Lambda}^s(\theta) &= S^{ss}(\theta) \\ \hat{\Lambda}^u(\theta) &= S^{uu}(\theta) \\ \hat{\Lambda}^t(\theta) &= D\hat{f}(\theta) \end{aligned}$

with a new error $\hat{S}(\theta) = DF(\hat{K}(\theta))\hat{P}(\theta) - \hat{P}(\hat{f}(\theta))\hat{\Lambda}(\theta)$ quadratically small than $S(\theta)$.

We have to repeat **STEP** 1 and **STEP 2** until new errors R and S achieve the desired error-tolerance.

The inverse of $P(\theta)$ as an unknown

Adding the inverse of $P(\theta)$ as another unknown $\rightsquigarrow$ make faster the method

Let $PI_0(\theta)$ as an approximation of $P_0^{-1}(\theta)$, with an error $EPI(\theta) = PI_0(\theta)P(\theta) - Id_{n\times n}$ small.

If the new approximation is $\hat{P}I(\theta) = PI_0(\theta) + QI(\theta)PI_0(\theta)$ with a modification $QI(\theta) = -E\hat{P}I(\theta)$, the improved inverse becomes:

$$\hat{P}I(\theta) = PI_0(\theta) - E\hat{P}I(\theta)PI_0(\theta)$$

which is computed each time after **STEP 2**.

Discretization

To apply this method numerically, we need the torus, bundles and dynamics discretizated.

We use the following discretization methods

- Interpolation
- Fourier expansions

This kind of discretization works correctly with this method as long as the invariant tori remains as a graph.

We consider the model manifold 1-dimensional, $\mathcal{M}^d = \mathbb{T}^1$, and we use functions $h(\theta)$ and $M(\theta)$ to represent the problem:

$$\begin{aligned} h &: \mathbb{T}^1 \to \mathbb{R}^n \\ M &: \mathbb{T}^1 \to M_{n \times n} \end{aligned}$$

Discretization by interpolation

We will use functions h represented as a Lagrange Interpolating Polynomial of degree r:

$$h(\tau) = \sum_{i=0}^{r} h(\theta_i)L_i(\tau)$$

So we need to storage a finite mesh of values over the function.

Let h discretizated by function values $h(\theta_j)$ at N equidistant meshing points $\theta_j \in \mathbb{T}^1$, $j = 1, \dots, N$, N large enough to well approximate h.

When we need the value $h(\tau)$, for some $\tau \neq \theta_j$, $j = 1, \dots, N$ we have to:

1. Look for θ_j and θ_{j+1} such that: $\dots < \theta_j \leq \tau < \theta_{j+1} < \dots$.
2. Compute the *Lagrange basis polynomials* of this value τ

$$L_i(\theta) = \frac{\prod_{j=0, j\neq i}^{r} \theta - \theta_j}{\prod_{j=0, j\neq i}^{r} \theta_i - \theta_j}, \qquad i = 0, \dots, r$$

3. Compute *Lagrange Polynomial* of h on τ:

$$h(\tau) = \left(\sum_{i=0}^{r} h(\theta_i) L_i(\tau) \right) \mod 2\pi$$

Lagrange basis polynomials modulus 2π

We are interpolating periodic functions modulus 2π instead of real functions $\rightsquigarrow$ TAKE CARE in the computation of Lagrange basis polynomials modulus 2π

We use the next convention (interpolation cubic case) to consider the nodes $\theta_{j-1} < \theta_j \leq \tau < \theta_{j+1} < \theta_{j+1}$:

$$\begin{array}{llllll} j=0 & \Rightarrow & \theta_{j-1} = \theta_{N-1} - 2\pi, & \theta_j = \theta_0, & \theta_{j+1} = \theta_1, & \theta_{j+1} = \theta_2 \\ j=N-2 & \Rightarrow & \theta_{j-1} = \theta_{N-3}, & \theta_j = \theta_{N-2}, & \theta_{j+1} = \theta_{N-1}, & \theta_{j+1} = \theta_0 + 2\pi \\ j=N-1 & \Rightarrow & \theta_{j-1} = \theta_{N-2}, & \theta_j = \theta_{N-1}, & \theta_{j+1} = \theta_0 + 2\pi, & \theta_{j+1} = \theta_1 + 2\pi \end{array}$$

Otherwise, we have all values inside $(0, 2\pi)$ and we don't have any modulus conflicting problem.

To compute the tangent bundle and tangent linearized dynamics, as we need to interpolate derivatives of K and f, we may compute the *derived Lagrange basis polynomials* instead of the Lagrange basis polynomials, and other computations are identically the same.

Discretization by Fourier expansion

We will use functions h represented as Fourier expansions

$$h(\tau) = c_0 + \sum_{k=0}^{r} c_k \cos(k\tau) + s_k \sin(k\tau)$$

Then, we need to storage a finite number r of Fourier coefficients s_k, c_k for each function we need in the method.

With this discretization method, we could add a condition to check that the approximations we found are good approximated by inspecting the tails of the Fourier series.

Implementation

Continue K NHIT with respect one parameter

- The algorithm is applied to continuation of invariant curves, saddle or attractor ones.
- We continue the torus w.r.t. ONE parameter regardless its dynamics, crossing resonances.
- We explore the mechanism of breakdown of the saddle invariant curve.

Continuation steps

- Let our discrete dynamical system F depends on one perturbation parameter ϵ.
- For some ϵ_0, suppose known (or could had been computed explicitly) a K NHIT with his dynamics and bundles.
- Varying ϵ, we apply the method to compute a new K, f, P, Λ for each new ϵ.
- We can follow incrementing ϵ while the method converges for new epsilon: while we reach the prefixed tolerance in less than 15 steps. At this moment, the invariant torus breakdown (it ceases to be normally hyperbolic).

3D Fattened Arnold Family

The Fattened Arnold Family 3-dimensional (3D-FAF) is a map $F : \mathbb{T}^1 \times \mathbb{R}^2 \to \mathbb{T}^1 \times \mathbb{R}^2$ defined by :

$$F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + a + \epsilon(\sin(x) + y + z/2) \\ b(\sin(x) + y) \\ c(\sin(x) + y + z) \end{pmatrix}$$

where $a \in \mathbb{T}^1$, b,$c \in \mathbb{R}$ are fixed parameters ($b < 1$, $c > 1$) and $\epsilon \in \mathbb{R}$ is the perturbation parameter.

We apply our method with $\mathcal{M}^1 = \mathbb{T}^1$ and $n = 3$.

We use the fixed parameters fixed as:

$$\begin{aligned} a &= 0.1 \\ b &= 0.3 \\ c &= 2.4 \end{aligned}$$

Unperturbed case: $\epsilon = 0$

$\rightsquigarrow$ $a = 0$: we have an explicit invariant circle given by the expression:

$$x \in \mathbb{T}^1 \ : \ K_0(x) = \left(x, \frac{b}{1-b}\sin(x), \frac{c}{(1-b)(1-c)}\sin(x) \right) \quad (2)$$

which is a graph.

$\rightsquigarrow$ $a > 0$: an invariant circle already exists and it is an approximation of (2), of the form:

$$K_0(x) = (x, \varphi(x), \psi(x)), \quad x \in \mathbb{T}^1$$

By invariance, it has to meet the invariance equation:

$$F((x, \varphi(x), \psi(x))) = (f_0(x), \varphi(f_0(x)), \psi(f_0(x)))$$

from where we obtain the initial dynamics: $f_0(x) = x + a$.

Unperturbed case: $\epsilon = 0$

- Second coordinate equality of the equation is a contraction (as $b = 0.3 < 1$)
- Third coordinate equality of the equation is an expansion (as $c = 2.4 > 1$)

So, we could solve each equality by iteration obtaining the initial approximation of the invariant circle:

$$K_0(x) = (x, \varphi(x), \psi(x)), \text{ where } \begin{cases} x \in \mathbb{T}^1 \\ \varphi(x) = \sum_{k=1}^{\infty} b^k \sin(x - ka) \\ \psi(x) = -\sum_{k=1}^{\infty} \frac{\varphi(x+ka)+\sin(x+ka)}{c^k} \end{cases}$$

Perturbed case: $\epsilon > 0$

- It hasn't fixed points while $\epsilon \in [0, \epsilon_1)$, $\epsilon_1 := \left| \frac{-2a(1-b)(1-c)}{2-c} \right|$
- At ϵ_1 a fixed point appears (have a **FOLD**) with eigenvalues:

$$\begin{aligned} \lambda_1 &= c > 1 \\ \lambda_2 &= 1 \\ \lambda_3 &= b < 1 \end{aligned}$$

- For $\epsilon > \epsilon_1$, this fixed point splits into two saddles, with eigenvalues:

SADDLE 1			SADDLE 2		
$1 <$	λ_1	$< c$	$1 <$	c	$< \mu_1$
$b <$	λ_2	< 1	$1 <$	μ_2	$< c$
$b <$	λ_3	< 1	$\mu_3 <$	b	< 1

We can increase ϵ until $\epsilon_2 :\approx 0.776177304$, where λ_2 and λ_3 collide (with $\lambda \approx 0.624$) and become two complex conjugate eigenvalues. At this point, the invariant circle loss its normal hyperbolicity.

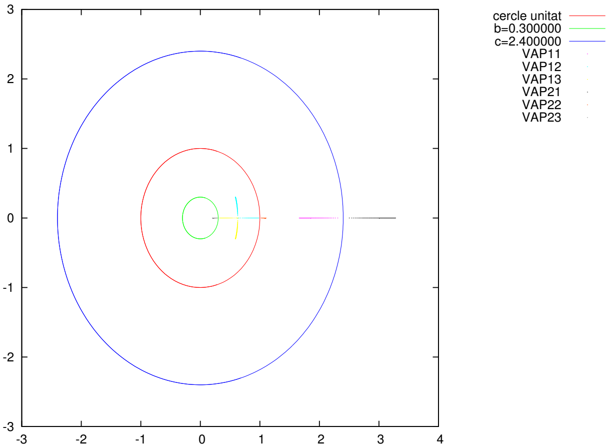


Figure: Evolution of all eigenvalues for the two fixed points of 3D-FAF from fold $\epsilon_1 = 0.49$ until $\epsilon = 1$.

First initial approximation

We use as a initial approximation:

- $K_0(\theta) = \left(\theta, \sum_{k=1}^{10^5} b^k \sin(\theta - ka), -\sum_{k=1}^{10^5} \frac{\varphi(\theta+ka)+\sin(\theta+ka)}{c^k} \right)$

- $f_0(\theta) = \theta + a$

- $\Lambda_0(\theta) = \begin{pmatrix} \Lambda_0^t(\theta) & 0 & 0 \\ 0 & \Lambda_0^s(\theta) & 0 \\ 0 & 0 & \Lambda_0^u(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$
  the three eigenvalues of DF_0.

- $P_0(\theta) = \left(\frac{\partial K_0}{\partial \theta}(\theta) \;\middle|\; N_0^s(\theta) \;\middle|\; N_0^u(\theta) \right)$, where

$$N_0^s(\theta) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } N_0^u(\theta) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We use the method with:

- INTERPOLATION DISCRETIZATION CASE: initial mesh of the parameterization of torus: 200 points.
- FOURIER DISCRETIZATION CASE: initial number of Fourier coefficients: 50 coefficients .
- As a tolerance of the errors: $||R||, ||S||, ||EPI|| < 10^{-10}$.

At the end, we obtain:

$\rightsquigarrow$ INTERPOLATION DISCRETIZATION CASE:

- final mesh of the parameterization of torus: 409.600 points.
- final epsilon value: 0.7418701172.

$\rightsquigarrow$ FOURIER DISCRETIZATION CASE:

- final number of Fourier coefficients: 10.000 coefficients.
- final epsilon value: 0.7166513681.

INTERPOLATION DISCRETIZATION CASE

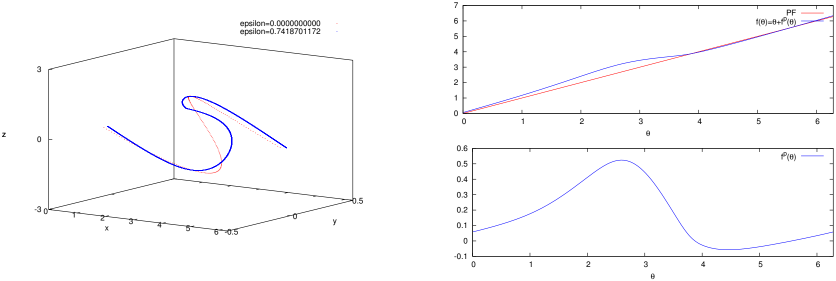
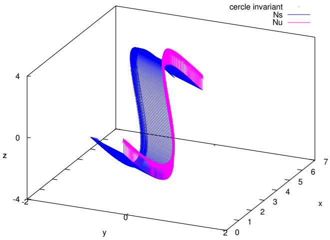
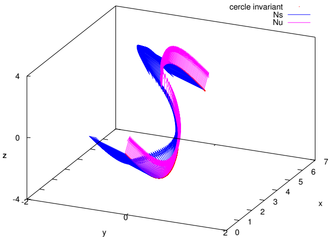


Figure: Evolution of the invariant tori (in red there the corresponding to the unperturbed system and in blue the last NHIM we can compute, for $\epsilon = 0.7418701172$) and the dynamics over our last tori.

INTERPOLATION DISCRETIZATION CASE



(a)



(b)

Figure: Variation of normal fibers from the unperturbed system (a) until the loss of nomal hyperbolicity, at $\epsilon = 0.7418701172$ (b)

INTERPOLATION DISCRETIZATION CASE

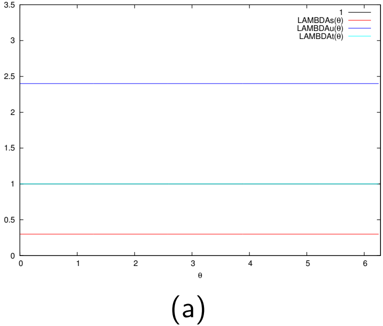
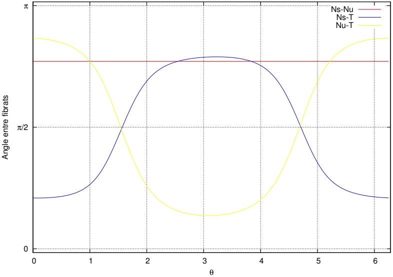
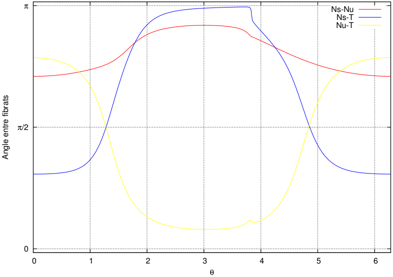


Figure: Evolution of the dynamics of tangent and normal bundle, from the unperturbed system (a) until $\epsilon = 0.7418701172$ (b).

INTERPOLATION DISCRETIZATION CASE



(a)



(b)

Figure: Angles between all bundles, from the unperturbed system (a) until $\epsilon = 0.7418701172$ (b).

FOURIER DISCRETIZATION CASE

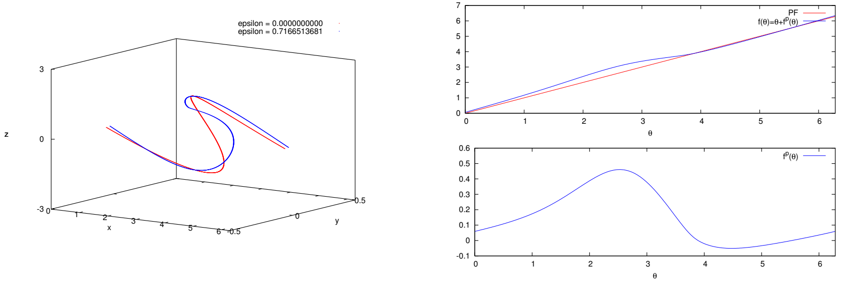
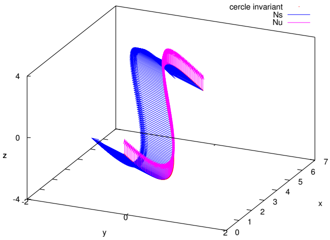
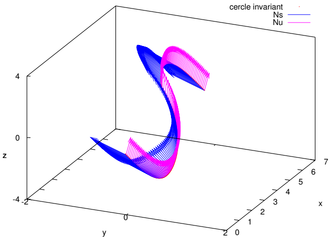


Figure: Evolution of the invariant tori (in red there the corresponding to the unperturbed system and in blue the last NHIM we can compute, for $\epsilon = 0.7166513681$) and the dynamics over our last tori.

FOURIER DISCRETIZATION CASE



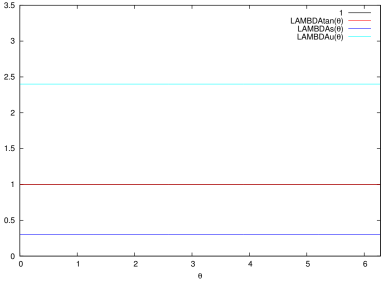
(a)



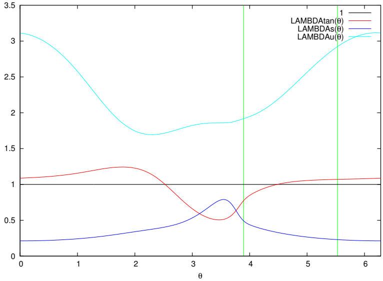
(b)

Figure: Variation of normal fibers from the unperturbed system (a) until the loss of normal hyperbolicity, at $\epsilon = 0.7166513681$ (b)

FOURIER DISCRETIZATION CASE



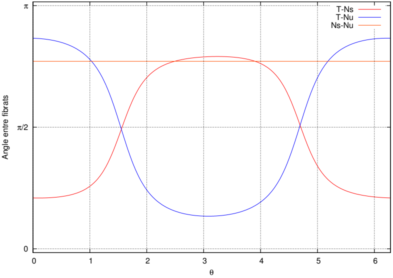
(a)



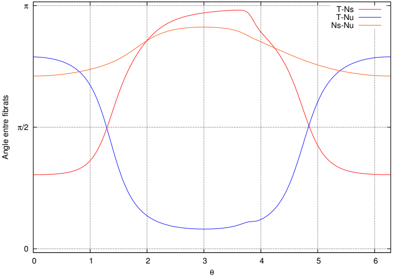
(b)

Figure: Evolution of the dynamics of tangent and normal bundle, from the unperturbed system (a) until $\epsilon = 0.7166513681$ (b).

FOURIER DISCRETIZATION CASE



(a)



(b)

Figure: Angles between all bundles, from the unperturbed system (a) until $\epsilon = 0.7166513681$ (b).

FOURIER DISCRETIZATION CASE

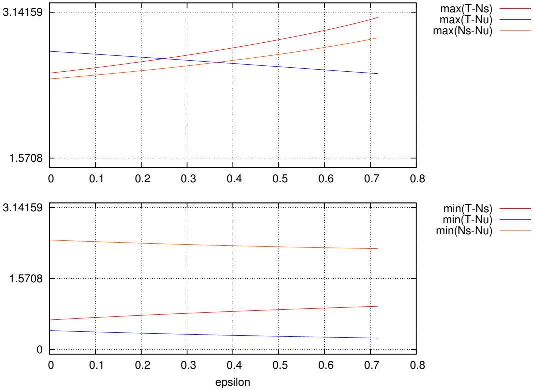


Figure: Maximum and minim angles between all bundles from $\epsilon = 0$ until $\epsilon = 0.7166513681$.