From the Earth to the Moon: the weak stability boundary and invariant manifolds

Priscilla A. Sousa Silva

MAiA-UB

Seminari Informal de Matemàtiques de Barcelona

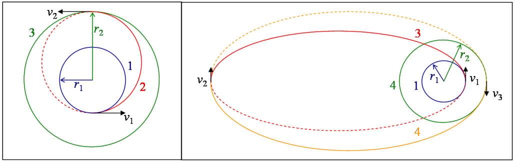
05-06-2012

Summary

Motivation

- Traditional techniques in astronautics (Hohmann transfer, bi-elliptic transfer)

- Hohmann to Mars and Venus: nearly the smallest possible amount of fuel, slow (8 months)
- Decades AND prohibitive amount of fuel to reach outer planets

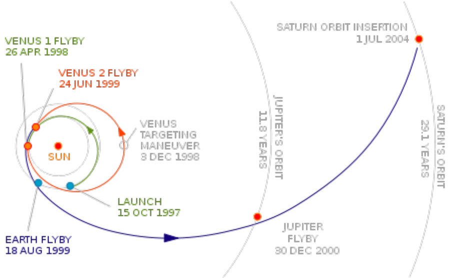


Hohmann transfer *Bi-elliptic transfer*

New challenges require new techniques!

- Patched conics: gravity assisted maneuvers to save fuel (swingby or gravitational slingshot)

Example: Cassini Mission, Oct 15, 1997 - Saturn multi-moon orbiter

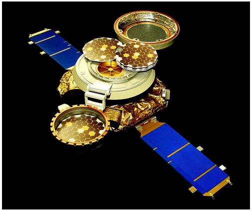


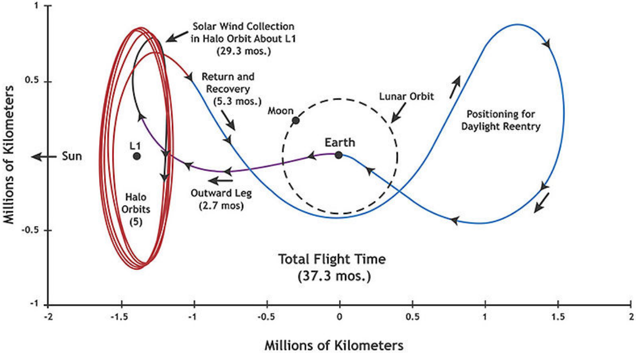
Motivation

Or...

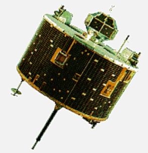
Take advantage of the fundamental dynamical structure of more realistic (N-body) models!

Example: Genesis Mission, Aug 8, 2001 - approximate heteroclinic return orbit





Context

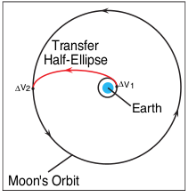


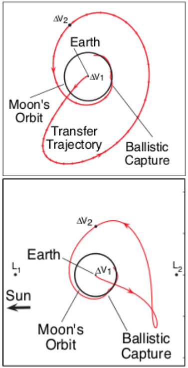
- Space mission projects based many-body dynamics: particularly Sun-Earth-Moon-Sc.
- WSB concept proposed heuristically by E. Belbruno (1987) related to Earth-Moon transfers with ballistic capture.
- Employed successfully in the rescue of the Japanese spacecraft Hiten in 1991 (Belbruno and Miller, 1990).

"Regions in the phase space where the perturbative effects of the Earth-Moon-Sun acting on the spacecraft tend to balance". (Belbruno and Miller, 1993)

"A location near the Moon where the spacecraft lies in the transition between ballistic capture and ejection". (Belbruno *et al.*, 2008)

But... Precise definition? Why does it work? How to find WSB trajectories?





The Restricted Three-body Problem

Equations of motion of P_3

$$\ddot{x} - 2\dot{y} = \Omega_x, \qquad \ddot{y} + 2\dot{x} = \Omega_y,$$

with $\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}$,

$r_1^2 = (x-\mu)^2 + y^2$, and $r_2^2 = (x+1-\mu)^2 + y^2$.

The integral of motion

$$J(x, y, \dot{x}, \dot{y}) = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2) = C, \quad C \text{ is the Jacobi constant.}$$

$$\mathcal{M}(\mu, C) = \left\{(x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 | J(x, y, \dot{x}, \dot{y}) = constant\right\}.$$

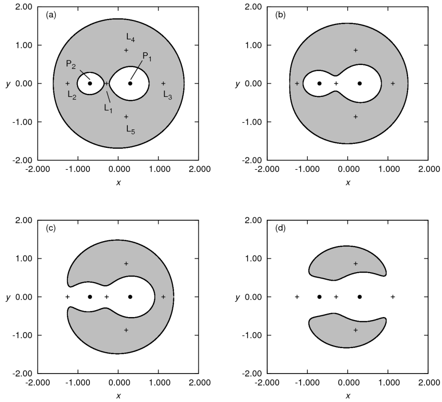
Equilibrium points

- $L_{1,2,3}$: collinear points, saddle-center.
- $L_{4,5}$: triangular points, stable if $m_1/m_2 > 24.96$.
- The Jacobi constant values evaluated at L_K are denoted by C_k, $k = 1, 2, 3, 4, 5$.

The Restricted Three-body Problem

Hill regions, $\mathcal{H}$

- Accessible areas for each C: $\mathcal{H}(\mu, C) = \{(x, y) | \Omega(x, y) \geqslant C/2\}$
- Bounded by the *zero-velocity curves*

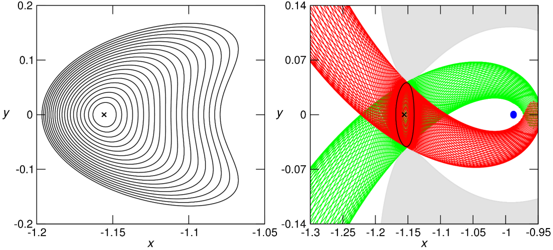


For a given μ, there are five different configurations for $\mathcal{H}$:

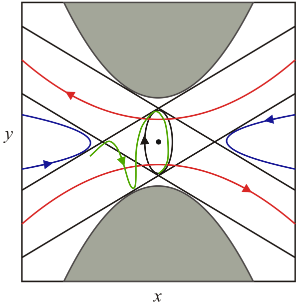
Case 1: $C > C_1$;
Case 2: $C_1 > C > C_2$;
Case 3: $C_2 > C > C_3$;
Case 4: $C_3 > C > C_4 = C_5$;
Case 5: $C_4 = C_5 > C$ - motion over the entire x-y plane is possible.

The Restricted Three-body Problem

Lyapunov Orbits



- Moser (1958) and Conley (1968,1969): existence of unstable periodic orbits around the collinear equilibria.



- Types of solution around the equilibria: **periodic**, **transit**, **asymptotic** and **non-transit**.
- Stable manifold (green): $W^s(\Gamma) = \{\mathbf{x} \in \mathbb{R}^4 : \phi(\mathbf{x}, t) \to \Gamma, t \to \infty\}$;
- Unstable manifold (red): $W^u(\Gamma) = \{\mathbf{x} \in \mathbb{R}^4 : \phi(\mathbf{x}, t) \to \Gamma, t \to -\infty\}$.

W^s and W^u are locally homeomorphic to 2D cylinders and act as separatrices of the phase space.

The algorithmic WSB: capture

Definition (Permanent capture: geometric concept)

P_3 is permanently captured into the P_1-P_2 system in forward (backward) time if $|\mathbf{q}|$ is bounded as $t \to \infty$ ($t \to -\infty$), and $|\mathbf{q}| \to \infty$ when $t \to -\infty$ ($t \to \infty$).

Definition (Temporary capture: geometric concept)

P_3 has temporary capture at $t = t^*, |t^*| < \infty$, if $|q(t^*)| < \infty$ and $\lim_{t\to\pm\infty} |q(t)| = \infty$.

But...

Definition (Ballistic capture: analytic concept - see Belbruno (2004))

P_3 is ballistically captured by P_2 at time $t = t_c$ if, for a solution $\varphi(t) = (x(t), y(t), \dot{x}(t), \dot{y}(t))$ of the R3BP, $h_K(\varphi(t_c)) \leq 0$, where h_K is the two-body energy of P_3 with respect to P_2.

Recall: $h_K = \frac{1}{2}(\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2) - \frac{Gm_2}{\tilde{r}_2}$, where $(\tilde{x}, \tilde{y}, \dot{\tilde{x}}, \dot{\tilde{y}})$ is the state of P_3 in an inertial reference frame with origin in P_2.

Note: usage of *ballistic* not unique! e.g., a situation in which no propulsion is needed to achieve temporary capture - see Koon et al (2001); Marsden and Ross (2005).

Weak Stability Boundary Algorithmically Defined

Construction

García and Gómez (2007): Consider a radial segment $l(\theta)$ departing from the smaller primary P_2 and making an angle θ with the x-axis. Take trajectories for P_3, starting on $l(\theta)$ such that:

- P_3 starts its motion on the periapsis of an osculating ellipse around P_2 ($r_2 = a(1-e)$).
- The eccentricity of the initial Keplerian motion is kept constant along $l(\theta)$.
- The initial velocity vector of the trajectory is perpendicular to $l(\theta)$. The modulus of the initial velocity is $\nu^2 = \mu(1+e)/r_2$.
- The initial two-body Kepler energy of P_3 w.r.t. P_2 is negative, i.e., $e \in [0,1)$, since the Kepler energy computed at the periapsis is $h_K = \mu(e-1)/(2r_2)$.

Initial conditions for the motion

- Clockwise (positive) osculating motions

$$\begin{aligned} x &= -1+\mu+r_2\cos\theta, & y &= r_2\sin\theta, \\ \dot{x} &= r_2\sin\theta - \nu\sin\theta, & \dot{y} &= -r_2\cos\theta + \nu\cos\theta, \end{aligned}$$

- Counterclockwise (negative) osculating motions

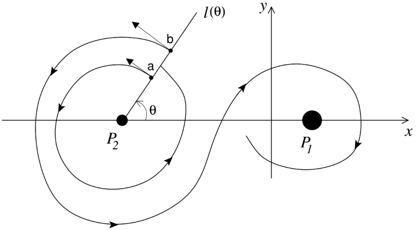
$$\begin{aligned} x &= -1+\mu+r_2\cos\theta, & y &= r_2\sin\theta, \\ \dot{x} &= r_2\sin\theta + \nu\sin\theta, & \dot{y} &= -r_2\cos\theta - \nu\cos\theta. \end{aligned}$$

Weak Stability Boundary Algorithmically Defined

Stability classification

Definition (Stability)

The motion of P_3 is said to be **stable** *if after leaving $l(\theta)$ it makes a full cycle about P_2 without going around P_1 and returns to $l(\theta)$ with $h_K < 0$. The motion is* **unstable** *otherwise.*



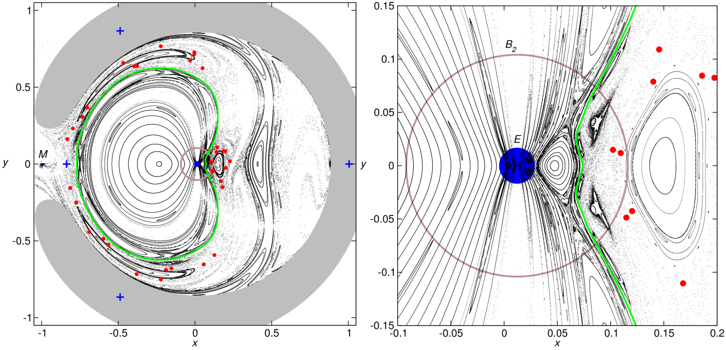
Definition (Algorithmic WSB)

The Weak Stability Boundary is given by the set $\partial\mathcal{W} = \{r^* | \theta \in [0, 2\pi), e \in [0, 1)\}$, where $r^*(\theta, e)$ are the points along the radial line $l(\theta)$ for which there is a change of stability. The subset obtained by fixing the eccentricity e of the osculating ellipse is $\partial\mathcal{W}^e = \{r^* | \theta \in [0, 2\pi), e = \text{constant}\}$.

Grid dependence! Integration time dependence!

Inner Transfers: within the restricted three-body problem

- Scheme to design low energy "periodic" Earth-to-Moon transfers.
  (i) the cost per cycle should be as small as is practical;
  (ii) control and stability problems should be as easy as possible;
  (iii) as much flexibility should be build into the scheme as possible.
  Conley C (1968) *Low energy transit orbits in the restricted three-body problem*. SIAM Journal of Applied Mathematics, v. 16, p. 732-746
- Impossibility!!!
  McGehee R (1969) *Some homoclinic orbits for the restricted three-body problem*. PhD thesis, University of Wisconsin, Madison.



Outer transfers

■ **Four body models required to obtain assist by the Sun!**

Patched Three-Body approach

- Sun-Earth-Moon system approximated by:

$$\text{Sun-Earth-SC } (SE \Leftrightarrow \odot) + \text{Earth-Moon-SC } (EM \Leftrightarrow \oplus).$$

- Complete transfer orbit o_c (departing from a LEO o_i and arriving at a LLO o_f):

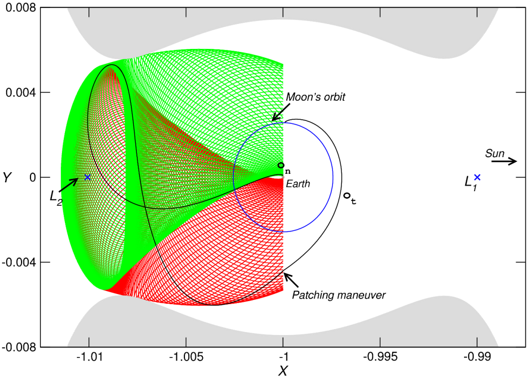
$$\text{non-transit orbit } o_n \text{ associated to } L_1^{\odot} \text{ or } L_2^{\odot} + \text{transit orbit } o_t \text{ associated to } L_2^{\oplus}.$$

- Total energy: Δv_1 to leave o_i + Δv_2 at patching point + Δv_3 to enter o_f.

Differential correction needed to obtain final solution!

Patched Three-Body approach

Complete orbit (o_c)

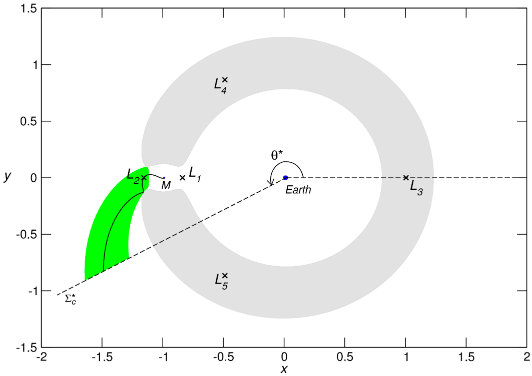


o_n: SE leg
o_t: EM leg

x-y projection of the inner branches $W^s_\odot$ (green) and $W^u_\odot$ (red) of $\Gamma_2^\odot(C^\odot = 3.00080369)$

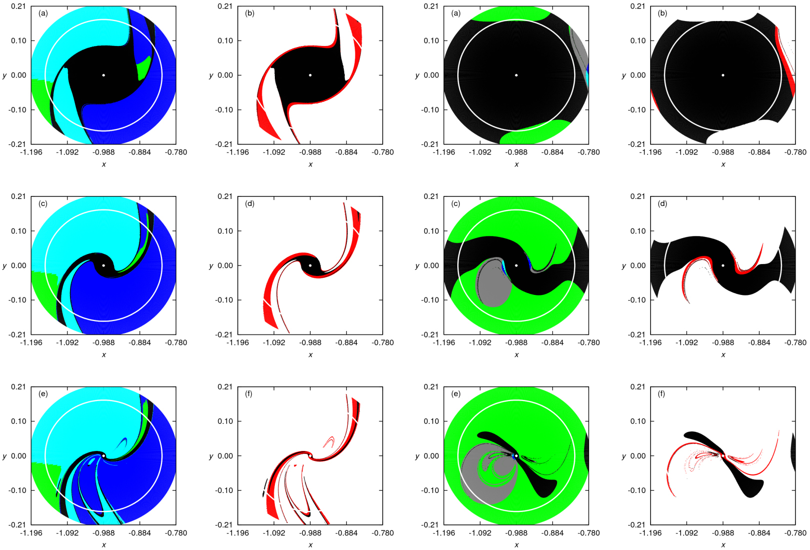
Focus on the Earth-Moon leg

■ **Hiten-like??? Ballistic capture??? But using manifold structure!**



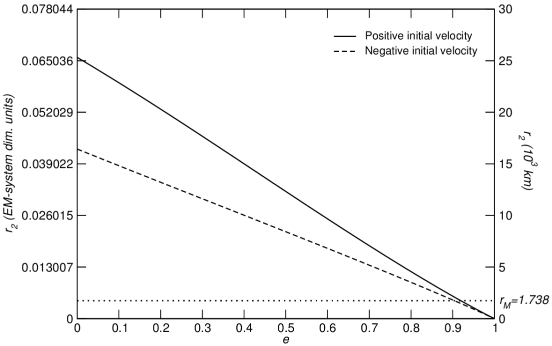
- Koon W, Lo M, Marsden J, Ross S (2000) Shoot the Moon. In: Proceedings of AAS/AIAA Space Flight Mechanics Meeting, AAS 00-166
- Koon W, Lo M, Marsden J, Ross S (2001) Low energy transfer to the Moon. Celestial Mechanics and Dynamical Astronomy 81, p. 63-73
- Sousa Silva P (2011) The algorithmic WSB in Earth-to-Moon mission design: dynamical aspects and applicability. PhD thesis, Instituto Tecnológico de Aeronáutica - São José dos Campos

Implementation of the algorithmic WSB



Preliminary checks: the energy

High excentricity needed to allow low capture orbits!

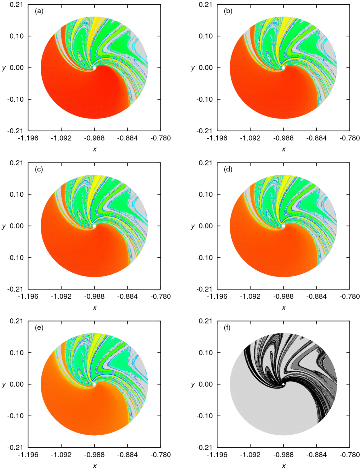


$$C(r_2,\theta,e) = (1-\mu) + \frac{2\mu}{r_2} - 2(1-\mu)r_2\cos(\theta) + r_2^2 + \frac{2(1-\mu)}{\sqrt{1-2r_2\cos(\theta)+r_2^2}} - \left[r_2 \mp \sqrt{\frac{\mu(1+e)}{r_2}}\right]^2$$

Energy gap between + and - sets of initial conditions: $\Delta C(r_2,\theta,e) = 4\sqrt{\mu(1+e)r_2}$.

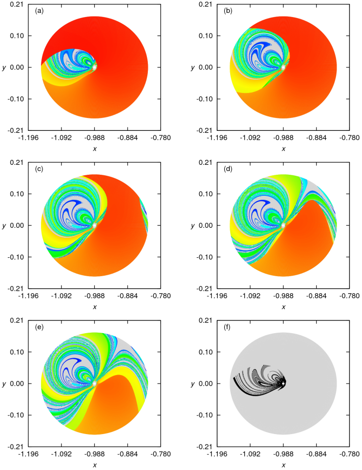
Checks for $t < 0$: applicability

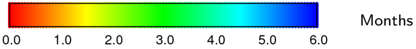
$IC^{+}(0.9)$



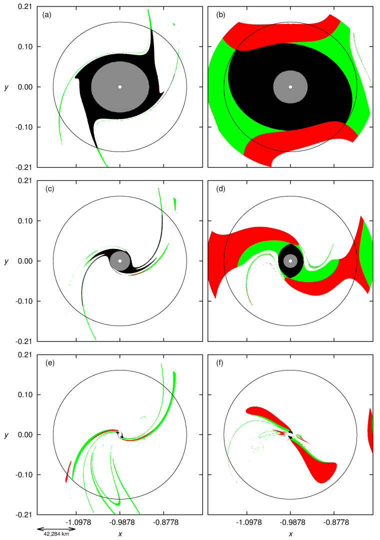
(a) Σ_1 **(b)** Σ_2 **(c)** Σ_3 **(d)** Σ_4 **(e)** Σ_5
(f) Col (black) / None (dark gray)

$IC^{-}(0.9)$

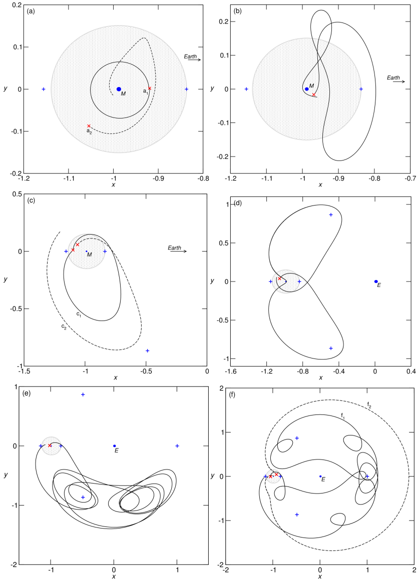




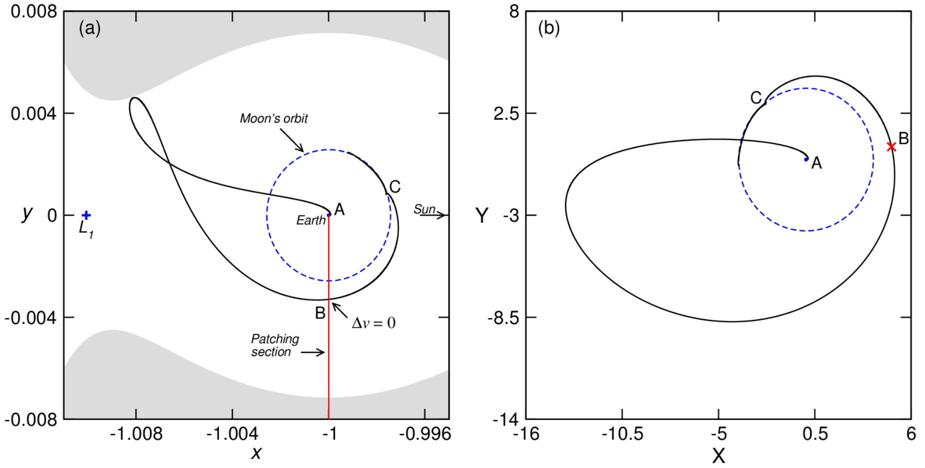
Checks for $t > 0$: stability



Red: $r_f > r_S$; Green: $r_f < r_S$; Black and Gray: $r_2(t) < r_S, \forall t \in [0, t_f]$, for $C < C_1$ and $C \geqslant C1$

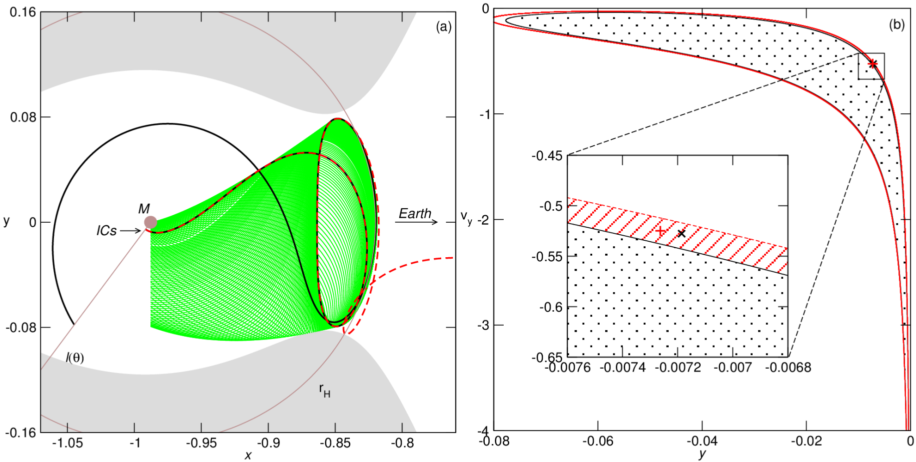


Checks for $t < 0$ + checks for $t > 0$ provide Earth-to-Moon transfers (within the Patched Three-body approach) with $\Delta v = 0$ at patching section!



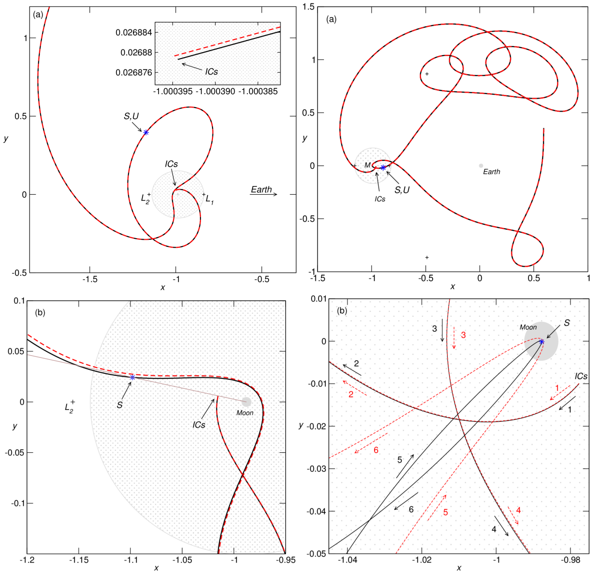
WSB corresponding to invariant manifols?

YES!

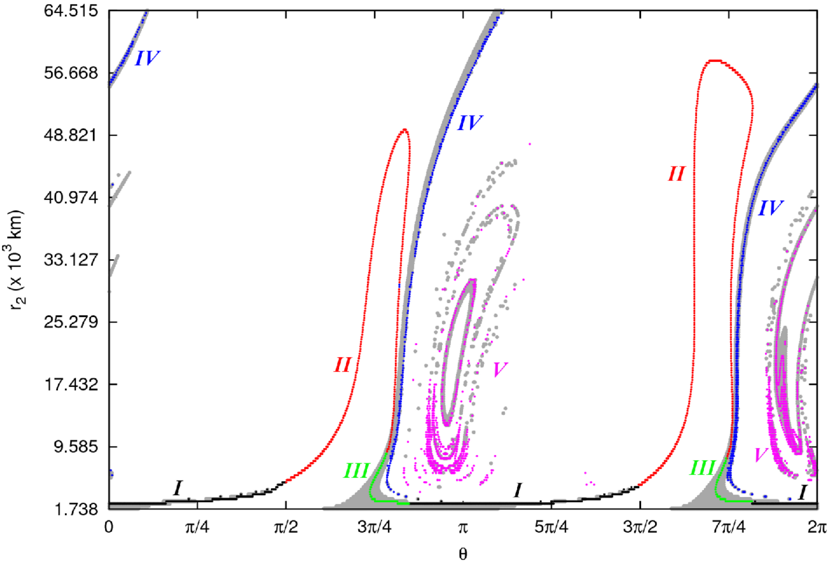


WSB corresponding to invariant manifols?

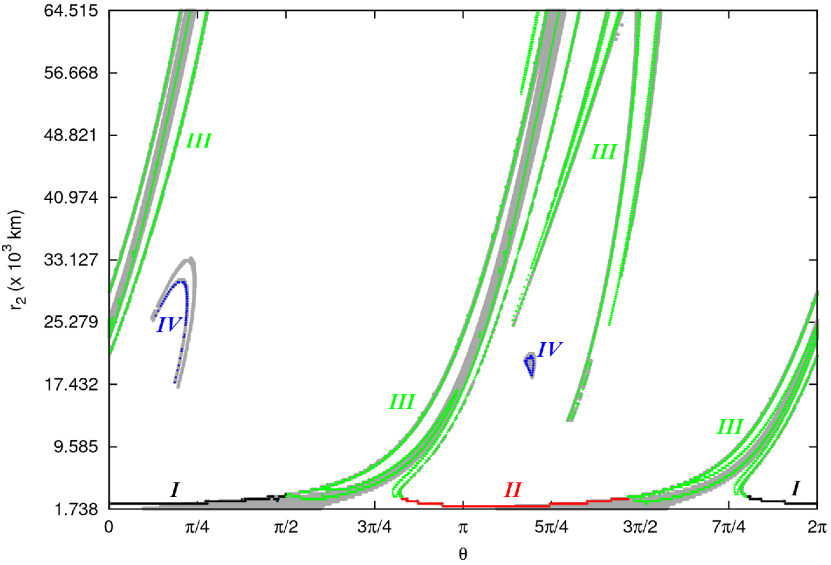
NO!



WSB corresponding to invariant manifols: up to which extent?



WSB corresponding to invariant manifols: up to which extent?



Final Remarks

- Need to develop new strategies for mission design $\Rightarrow$ use the rich dynamical structure in systems of many bodies
- Q. Why to study WSB-like approaches if we have manifold structure?
  A. Manifold structure not always avaliable!
  A. Manifold structure too complicated to be easily obtained!
- The algorithmic WSB: needs revision...
  Q. What has been done is this direction?
- Still... Provides (under some assumptions!) adequate initial conditions for Earth-to-Moon transfer orbits in the patched-three body approach with ZERO Δv at patching!