Beurling-Landau's density on compact manifolds

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The Paley-Wiener space

- Paley-Wiener space,
  $PW^2_{[-\pi,\pi]} = \left\{ f \in L^2(\mathbb{R});\ \mathrm{supp}(\hat{f}) \subset [-\pi,\pi] \right\}.$

- $\Lambda = \{\lambda_n\}_n \subset \mathbb{R}$ is a **sampling sequence** if there exist constants $0 < A \leq B$ such that

$$A\|f\|_2^2 \leq \sum_n |f(\lambda_n)|^2 \leq B\|f\|_2^2, \quad \forall f \in PW^2_{[-\pi,\pi]}.$$

- $\Lambda = \{\lambda_n\}_n \subset \mathbb{R}$ is an **interpolating sequence** if for all $\{c_n\}_n \in \ell^2$ $\exists f \in PW^2_{[-\pi,\pi]}$ such that

$$f(\lambda_n) = c_n \quad \forall n.$$

Theorem (Whittaker-Shannon-Kotelnikov, 1915-1949-1933)

For all $f \in PW^2_{[-\pi,\pi]}$,

$$f(x) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)}.$$

- $\Lambda = \mathbb{Z}$ is sampling and interpolating for $PW^2_{[-\pi,\pi]}$.
- We can even take $\Lambda = \{\lambda_n\}_n$ with $\sup_n |\lambda_n - n| < \delta$ (Sharp $\delta = 1/4$).
- General cases: PW^2_E, with $E \subset \mathbb{R}^m$ bounded set. Landau (in 1967) proved necessary conditions for interpolation or sampling in terms of Beurling-Landau densities.

Sampling and Interpolation in $PW^2_{[-\pi,\pi]}$

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A compact setting like Paley-Wiener: $\mathbb{S}^1$

$$\mathcal{P}_n = \left\{ q = \sum_{k=0}^{n} a_k z^k,\ z \in \mathbb{S}^1 \right\}.$$

- $q \in \mathcal{P}_n$ play the role of bandlimited functions with bandwidth n.
- Sampling and Interpolation for these spaces: instead of sequences we shall take families $\mathcal{Z} = \{\mathcal{Z}(n)\}_n$, $\mathcal{Z}(n) = \{z_{nj}\}_{j=1}^{m_n} \subset \mathbb{S}^1$.

Paley-Wiener setting	The compact setting
$\Lambda = \{\lambda_n\}_n$ separated:	$\mathcal{Z} = \{z_{nj}\}$ separated:
$\delta := \inf_{n \neq m} \lvert\lambda_n - \lambda_m\rvert > 0$	$d(z_{nj}, z_{nk}) \geq \frac{\delta}{n},\ \forall j \neq k\ \forall n$

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Sampling in the compact setting

In the Paley-Wiener setting, Λ is sampling:

$$\|f\|_2^2 \simeq \sum_n |f(\lambda_n)|^2, \quad \forall f \in PW^2_{[-\pi,\pi]}.$$

In the compact setting, $\mathcal{Z}$ is **Marcinkiewicz-Zygmund (M-Z)** (or sampling):

$$\int_0^{2\pi} |q(e^{i\theta})|^2 d\theta \simeq \frac{1}{n} \sum_{j=1}^{m_n} |q(z_{nj})|^2, \quad \forall n, q \in \mathcal{P}_n.$$

Characterization for $\mathbb{S}^1$

Theorem (J.Ortega-Cerdà, J. Saludes, 2006)

Let $\mathcal{Z}$ be a separated family, i.e.

$$d(z_{nj}, z_{nk}) \geq \frac{C}{n}, \quad \forall j \neq k, n \in \mathbb{N}.$$

If

$$D^{-}(\mathcal{Z}) > \frac{1}{2\pi},$$

then $\mathcal{Z}$ is a M-Z family. Conversely, if $\mathcal{Z}$ is M-Z then $D^{-}(\mathcal{Z}) \geq \frac{1}{2\pi}$.

- J. Marzo proved necessary density conditions for $\mathbb{S}^m$.

Our setting

Notation for our setting

1. (M, g) is a smooth compact Riemannian manifold of dimension $m \geq 2$ without boundary.
2. The Laplacian on M is defined as:

$$\Delta_g(f) = \frac{1}{\sqrt{|g|}} \sum_i \partial_{x_i} \sum_j \sqrt{|g|} g^{ij} \partial_{x_j} f.$$

3. Δ_g has discrete spectrum

$$0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \ldots \to \infty.$$

4. $L^2(M)$ decomposes in a direct sum of real functions $\phi_i \in \mathcal{C}^\infty(M)$ s.t. $\Delta_g \phi_i = -\lambda_i^2 \phi_i$.

Motivation

For $L \geq 1$, consider the spaces

$$E_L = \left\{ f \in L^2(M) : f = \sum_{i=1}^{k_L} \beta_i \phi_i,\ \lambda_{k_L} \leq L \right\}.$$

Motivation: The space E_L behaves like a space of polynomials of degree less than L.

Examples.

1. $M = \mathbb{S}^1$: $\phi_n = \cos(n\theta), \sin(n\theta)$ and E_n is the space of polynomials of degree less than n.
2. $M = \mathbb{S}^m$ $(m > 1)$: E_L is the space of spherical harmonics.

Bernstein inequality for E_L:

$$\|\nabla f_L\|_\infty \lesssim L \|f_L\|_\infty, \quad \forall f_L \in E_L.$$

Observation: We shall work with balls of radius r/L.

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Reproducing kernel

The reproducing kernel for E_L is

$$K_L(z,w) := \sum_{i=1}^{k_L} \phi_i(z)\phi_i(w) = \sum_{\lambda_i \le L} \phi_i(z)\phi_i(w).$$

Hörmander (1968) has proved:

- $K_L(z,z) = C_m L^m + O(L^{m-1})$.
- $k_L = D_m L^m + O(L^{m-1})$, $k_L = \dim(E_L) = \#\{\lambda \le L\}$.

For L big enough:

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2. $\|K_L(z,\cdot)\|_2^2 = K_L(z,z) \simeq L^m \simeq k_L$.

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Interpolating and M-Z families on compact manifolds

Marcinkiewicz-Zygmund family

$\mathcal{Z} = \{\mathcal{Z}(L)\}_{L\geq 1}$ be a triangular family in M with $\mathcal{Z}(L) = \{z_{Lj}\}_{j=1}^{m_L}$.

Definition

$\mathcal{Z}$ is a ***Marcinkiewicz-Zygmund*** *(M-Z) family, if $\exists C > 0$ s.t.*
$\forall L \geq 1, f_L \in E_L$

$$\frac{C^{-1}}{k_L} \sum_{z_{Lj}\in\mathcal{Z}(L)} |f_L(z_{Lj})|^2 \leq \int_M |f_L|^2 dV \leq \frac{C}{k_L} \sum_{z_{Lj}\in\mathcal{Z}(L)} |f_L(z_{Lj})|^2.$$

Equivalently, $\mathcal{Z}$ is M-Z iff $\{K_L(z, z_{Lj})/\|K_L(\cdot, z_{Lj})\|\}$ form a frame, i.e.

$$\|f\|_2^2 \simeq \sum_{z_{Lj}\in\mathcal{Z}(L)} \left| \langle f, \frac{K_L(\cdot, z_{Lj})}{\sqrt{K_L(z_{Lj}, z_{Lj})}} \rangle \right|^2.$$

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Left hand side inequality

One inequality in M-Z is “easy”: usually called Plancherel-Pólya Theorem.

Theorem (J. Ortega-Cerdà, B. Pridhnani, 2010)

$\mathcal{Z}$ is a finite union of uniformly separated families iff $\exists C > 0$ s.t. $\forall L \geq 1$ and $f_L \in E_L$

$$\frac{1}{k_L}\sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \leq C \int_M |f_L(\xi)|^2 dV(\xi).$$

$\mathcal{Z}$ is **uniformly separated** if there exists a $\epsilon > 0$ s.t. for all $L \geq 1$

$$d_M(z_{Lj}, z_{Lk}) \geq \frac{\epsilon}{L}, \;\; j \neq k.$$

For M-Z families we need to study:

$$\int_M |f_L(\xi)|^2 dV(\xi) \lesssim \frac{1}{k_L}\sum_{j=1}^{m_L} |f_L(z_{Lj})|^2.$$

Interpolating family

Definition

$\mathcal{Z} = \{\mathcal{Z}(L)\}_{L\geq 1}$ ($m_L \leq k_L$) is an ***interpolating family*** *if for all $c = \{c_{Lj}\}_{L,1\leq j\leq m_L}$ s.t.*

$$\sup_{L\geq 1} \frac{1}{k_L} \sum_{j=1}^{m_L} |c_{Lj}|^2 < \infty,$$

there exists a $\{f_L\}_L$, $f_L \in E_L$ s.t. $\sup_{L\geq 1} \|f_L\|_2 < \infty$ and $f_L(z_{Lj}) = c_{Lj}$ ($1 \leq j \leq m_L$)

- $\mathcal{Z}$ interpolating $\Rightarrow$ $\mathcal{Z}$ is **uniformly separated**, i.e. there exists a $\epsilon > 0$ s.t. for all $L \geq 1$

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- $\mathcal{Z}$ separated enough $\Rightarrow$ $\mathcal{Z}$ interpolating.

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The Beurling-Landau density

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Let μ_L and σ be the normalized counting/volume measure, i.e.

$$d\mu_L = \frac{1}{k_L} \sum_{z_{Lj} \in \mathcal{Z}(L)} \delta_{z_{Lj}}, \quad d\sigma = dV/\mathrm{vol}(M).$$

The upper and lower density are:

$$D^+(\mathcal{Z}) = \limsup_{R\to\infty} \left(\limsup_{L\to\infty} \left(\max_{\xi\in M} \frac{\mu_L(B(\xi, R/L))}{\sigma(B(\xi, R/L))} \right) \right),$$

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Essentially, in the density we are looking at

$$\frac{\#(\mathcal{Z}(L) \cap B(\xi, R/L))}{R^m}.$$

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Main result

Theorem (J. Ortega-Cerdà, B. Pridhnani, 2010)

If $\mathcal{Z}$ is a M-Z family then there exists a unif. sep. M-Z family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ s.t.

$$D^{-}(\tilde{\mathcal{Z}}) \geq 1.$$

If $\mathcal{Z}$ is an interpolating family then it is unif. sep. and

$$D^{+}(\mathcal{Z}) \leq 1.$$

Concentration Operator

Definition

Let $A \subset M$. $\mathcal{K}_A : E_L \to E_L$ is defined as

$$\mathcal{K}_A f_L(z) = \int_A K_L(z,\xi) f_L(\xi) dV(\xi),$$

i.e.

$$\begin{aligned} \mathcal{K}_A : E_L &\longrightarrow L^2(M) \longrightarrow E_L \\ f_L &\longrightarrow \chi_A f_L \longrightarrow P_{E_L}(\chi_A f_L) \end{aligned}$$

$\mathcal{K}_A$ is self-adjoint $\Rightarrow$ Eigenvalues of $\mathcal{K}_A$ are real and E_L has an orthonormal basis of eigenvectors of $\mathcal{K}_A$.

Ingredients for the proof

- A localization property gives an estimate of $\#(\mathcal{Z}(L) \cap A)$ in terms of the "big eigenvalues" of $\mathcal{K}_A$.

- Relation of the eigenvalues of $\mathcal{K}_A$ with $\mathrm{tr}(\mathcal{K}_A)$ and $\mathrm{tr}(\mathcal{K}_A \circ \mathcal{K}_A)$.

- Trace estimates of $\mathcal{K}_A$ and $\mathcal{K}_A \circ \mathcal{K}_A$.

Notation. Eigenvalues of $\mathcal{K}_{A_L}$ ($A_L = B(\xi, R/L)$).

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Relation with the trace

General fact: Let T be an operator with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < 1$ and

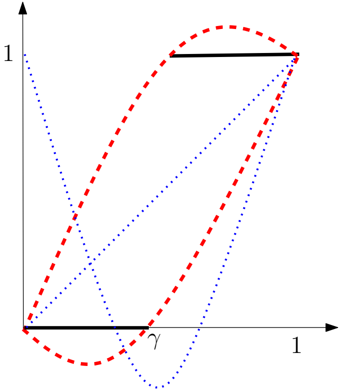
$$d\mu = \sum_{i=1}^{n} \delta_{\lambda_i}.$$

$$\mathrm{tr}(T) = \sum_{i=1}^{n} \lambda_i = \int_0^1 x d\mu(x),$$

$$\mathrm{tr}(T^2) = \sum_{i=1}^{n} \lambda_i^2 = \int_0^1 x^2 d\mu(x),$$

$$\#\{\lambda_i \geq \gamma\} = \int_\gamma^1 d\mu(x)$$

$$= \int_0^1 \chi_{[\gamma,1]}(x) d\mu(x).$$



Relation with the trace

In the M-Z case ($A_L = B(\xi, R/L)$):

$$\#(\mathcal{Z}(L) \cap A_L) \text{ “} \geq \text{”} \#\left\{\lambda_j^L > \gamma\right\} \geq \mathrm{tr}(\mathcal{K}_{A_L}) - \frac{\mathrm{tr}(\mathcal{K}_{A_L}) - \mathrm{tr}(\mathcal{K}_{A_L} \circ \mathcal{K}_{A_L})}{1-\gamma}.$$

$$\mathrm{tr}(\mathcal{K}_{A_L}) = \int_{A_L} K_L(z,z)\,dV(z) = k_L \frac{\mathrm{vol}(B(\xi, R/L))}{\mathrm{vol}(M)} + \frac{o(L^m)}{L^m}.$$

Recall that density is

$$\frac{\frac{1}{k_L}\#(\mathcal{Z}(L) \cap B(\xi, R/L))}{\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}} \left(\simeq \frac{\#(\mathcal{Z}(L) \cap B(\xi, R/L))}{R^m}\right).$$

$$\mathrm{tr}(\mathcal{K}_{A_L}) - \mathrm{tr}(\mathcal{K}_{A_L}^2) = \int_{A_L \times M \setminus A_L} |K_L(z,w)|^2 dV(w)\,dV(z).$$

We need $\limsup_{L\to\infty} \mathrm{tr}(\mathcal{K}_{A_L}) - \mathrm{tr}(\mathcal{K}_{A_L}^2) = o(R^m)$.

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Trace estimate

If the kernel has "good" bounds, i.e.

$$\lim_{L\to\infty}\int_{M\setminus B(\xi,r/L)}|\tilde{K}_L(z,\xi)|^2 dV(z) \lesssim \frac{1}{(1+r)^\alpha}, \quad \alpha\in(0,1), r\geq 1,$$

then everything works! (Maybe not true for all manifolds).

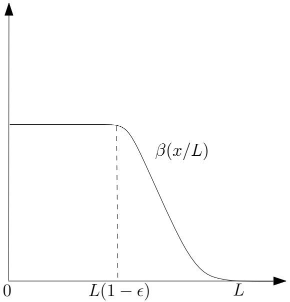
Examples: Compact Two-point homogeneous spaces: spheres, projective spaces.

For the general case: modify the concentration operator using "better" kernels.

$$B_L^{\epsilon}(z,w) = \sum_{i=1}^{k_L} \beta\left(\frac{\lambda_i}{L}\right)\phi_i(z)\phi_i(w),$$

and consider the transform from $L^2(M) \to E_L$

$$B_L^{\epsilon}(f)(z) = \int_M B_L^{\epsilon}(z,w)f(w)dV(w),$$



Nice bounds for these kernels (due to F. Filbir and H. Mhaskar).

Modified concentration operator

$$T_A^\epsilon f_L = B_L^\epsilon(\chi_A \cdot B_L^\epsilon)(f_L)$$

Landau's scheme used $\mathcal{K}_A f_L = P_{E_L}(\chi_A \cdot P_{E_L})(f_L)$.

Intuition: This operator is a smooth version of the classical concentration operator.

Now the trace estimates become:

$$\limsup_{L\to\infty} \left(\operatorname{tr}(T_{A_L}^\epsilon) - \operatorname{tr}(T_{A_L}^\epsilon \circ T_{A_L}^\epsilon)\right) \leq C_1(1-(1-\epsilon)^m)R^m + C_2R^{m-1},$$

where C_1 (indep. of ϵ) and C_2 are indep of R.

All error terms can be fixed.

Modified concentration operator

$$T_A^\epsilon f_L = B_L^\epsilon(\chi_A \cdot B_L^\epsilon)(f_L)$$

Landau's scheme used $\mathcal{K}_A f_L = P_{E_L}(\chi_A \cdot P_{E_L})(f_L)$.

Intuition: This operator is a smooth version of the classical concentration operator.

Now the trace estimates become:

$$\limsup_{L\to\infty} \left(\mathrm{tr}(T_{A_L}^\epsilon) - \mathrm{tr}(T_{A_L}^\epsilon \circ T_{A_L}^\epsilon)\right) \leq C_1(1-(1-\epsilon)^m)R^m + C_2 R^{m-1},$$

where C_1 (indep. of ϵ) and C_2 are indep of R.

All error terms can be fixed.

$$f \quad e^{i\frac{\pi}{2}} \, \mathbb{N}$$