Partial boundary value problems on finite networks

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Some definitions

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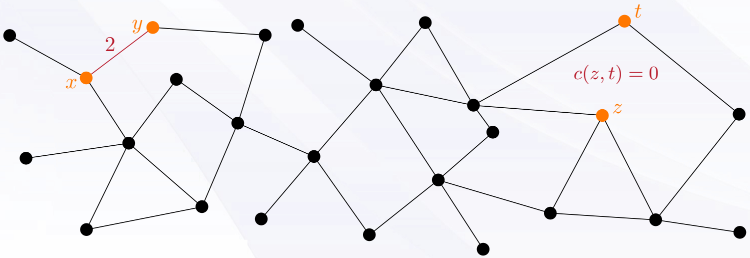
$\Gamma = (V, c)$ network, c conductances on the edges



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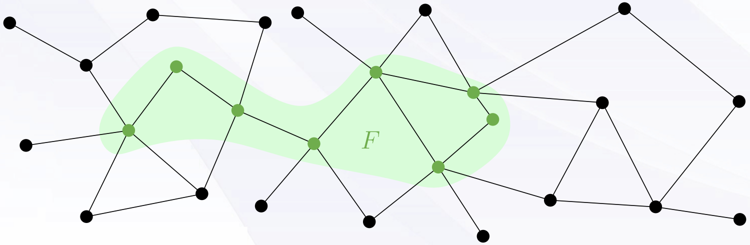
$F \subset V$ proper and connected subset, $\delta(F)$ boundary of F



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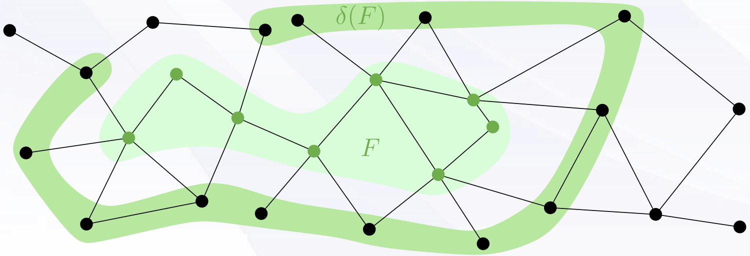

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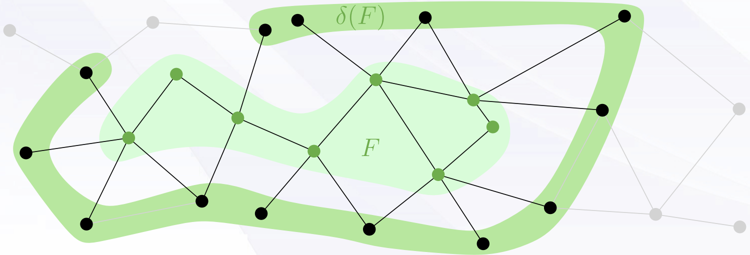

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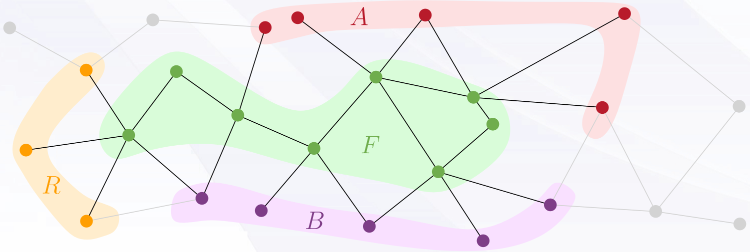
$F \subset V$ proper and connected subset, $\delta(F)$ boundary of F

$A, B \subset \delta(F)$ non-empty subsets, $A \cup B \neq \emptyset$

$R = \delta(F) \setminus (A \cup B)$

partition of the boundary



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⇝ We assume the network is in an electrical equilibrium state

⇝ We assume some information on the boundary to be known, as it can be phisically obtained from electrical boundary measurements

However, instead of having classical boundary information (simple information in all the boundary) we assume to have

R simple information

A double information

B no information at all!

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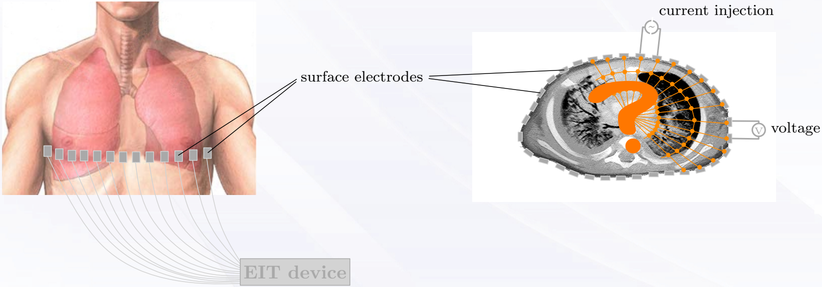
Our objective

- The Inverse BVPs arised in 1950 due to Calderón's work
- Medical purposes: *Electrical Impedance Tomography*



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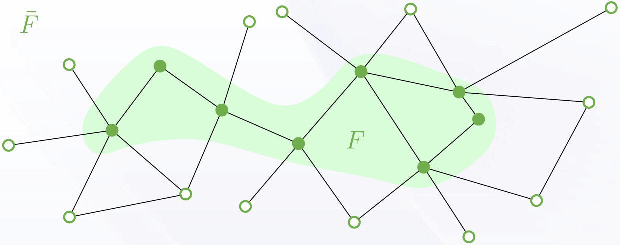

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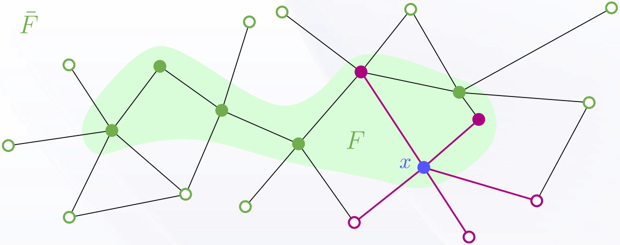

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$\mathcal{L} : \mathcal{C}(\overline{F}) \longrightarrow \mathcal{C}(\overline{F})$ Laplacian of Γ $u \in \mathcal{C}(\overline{F})$

$$x \in F \quad \rightsquigarrow \quad \mathcal{L}(u)(x) = \sum_{y \in \overline{F}} c(x,y)\big(u(x) - u(y)\big)$$

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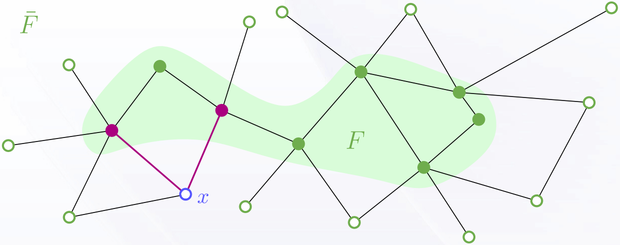

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$$x \in \delta(F) \rightsquigarrow \mathcal{L}(u)(x) = \sum_{y \in F} c(x,y)\big(u(x) - u(y)\big) = \frac{\partial u}{\partial \mathsf{n}_F}(x)$$

normal derivative

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Lemma (Bendito, Carmona, Encinas 2005)

$\mathcal{L}_q$ positive definite on $\mathcal{C}(F)$ $\Leftrightarrow$ there exists a weight ω such that $q \geq q_\omega$

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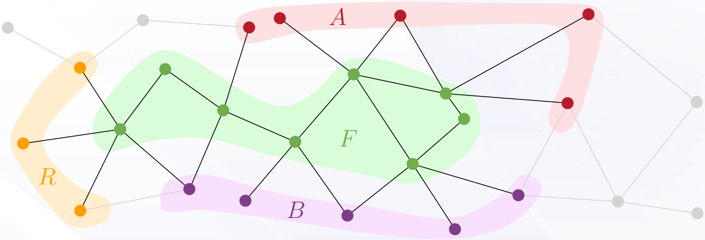
$\mathcal{L}_q$ positive definite on $\mathcal{C}(F)$ $\Leftrightarrow$ there exists a weight ω such that $q \geq q_\omega$

$\rightsquigarrow$ We work with potentials of the form $q = q_\omega + \lambda$, where $\lambda \geq 0$

Partial Dirichlet-Neumann boundary value problems

Partial Dirichlet-Neumann BVPs



Definition (Partial Dirichlet-Neumann BVP on F)

$$\begin{cases} \mathcal{L}_q(u) = h & \text{on } F \\ \dfrac{\partial u}{\partial \mathsf{n}_F} = g & \text{on } A \\ u = f & \text{on } A \cup R \end{cases}$$

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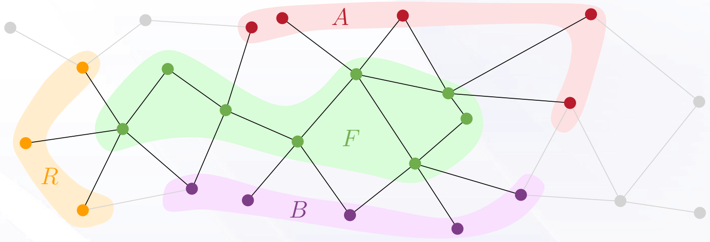
Partial Dirichlet-Neumann BVPs



Definition (*Homogeneous* partial Dirichlet-Neumann BVP)

$$\begin{cases} \mathcal{L}_q(u_h) = 0 & \text{on } F \\ \dfrac{\partial u_h}{\partial \mathsf{n}_{\mathsf{F}}} = 0 & \text{on } A \\ u_h = 0 & \text{on } A \cup R \end{cases}$$

its solutions are a vector subspace of $\mathcal{C}(F \cup B)$ that we denote by $\mathcal{V}_B$

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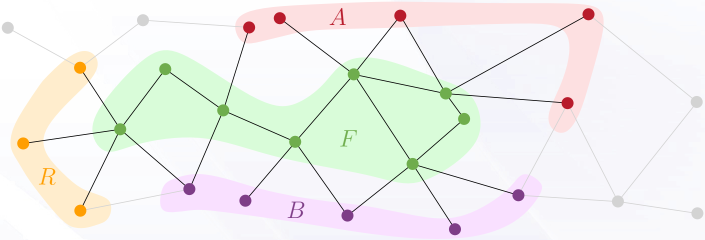

Definition (*Adjoint* partial Dirichlet-Neumann BVP)

$$\begin{cases} \mathcal{L}_q(u_a) = 0 & \text{on } F \\ \dfrac{\partial u_a}{\partial \mathsf{n}_F} = 0 & \text{on } B \\ u_a = 0 & \text{on } B \cup R \end{cases}$$

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Remember our partial BVP
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Remark

We need Green and Poisson operators!

Classical Green and Poisson operators

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⇝ The *classical Green operator* $\mathcal{G}_q$ solves the problem

$$\begin{cases} \mathcal{L}_q\Big(\mathcal{G}_q(h)\Big) = h & \text{on } F \\ \mathcal{G}_q(h) = 0 & \text{on } \delta(F) \end{cases}$$

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$\Rightarrow$ We need to modify these operators (we will see it later)

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Before modifying Green and Poisson operators, we need to define the Dirichlet-to-Neumann map as

$$\Lambda_q(g) = \frac{\partial \mathcal{P}_q(g)}{\partial \mathsf{n}_\mathsf{F}} \chi_{\delta(F)} \quad \text{for all} \quad g \in \mathcal{C}(\delta(F))$$

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that is, $DN_q(x,y) = \Lambda_q\big(\varepsilon_y\big)(x)$ for all $x, y \in \delta(F)$

Dirichlet-to-Neumann map - a little remark

Definition (Schur complement)

$A \in \mathcal{M}_{k\times k}(\mathbb{R})$, $B \in \mathcal{M}_{k\times l}(\mathbb{R})$, $C \in \mathcal{M}_{l\times k}(\mathbb{R})$ and $D \in \mathcal{M}_{l\times l}(\mathbb{R})$ with D non–singular

The Schur Complement of D on M, where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, is

$$^{M}\!/_{D} = A - BD^{-1}C \quad \in \mathcal{M}_{k\times k}(\mathbb{R})$$

Dirichlet-to-Neumann map - a little remark

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Theorem

The Dirichlet-to-Neumann map kernel $\mathsf{DN_q}$ *can be expressed as a Schur complement:*

$$\mathsf{DN_q}(\delta(\mathsf{F});\delta(\mathsf{F})) = {}^{\mathsf{L_q}(\bar{\mathsf{F}};\bar{\mathsf{F}})}\!/_{\mathsf{L_q}(\mathsf{F};\mathsf{F})}$$

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Corollary

If $P, Q \subseteq \delta(F)$, *then*

$$\mathsf{DN_q}(\mathsf{P};\mathsf{Q}) = {}^{\mathsf{L_q}(\mathsf{P}\cup\mathsf{F};\mathsf{Q}\cup\mathsf{F})}\!/_{\mathsf{L_q}(\mathsf{F};\mathsf{F})}$$

Modified Green and Poisson operators

Modified Green and Poisson operators

$$\begin{cases} \mathcal{L}_q(u) = h & \text{on } F \\ \dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = g & \text{on } A \\ u = f & \text{on } A \cup R \end{cases}$$

Using the Dirichlet-to-Neumann map, we can translate

Theorem

$$|A| - |B| = \dim \mathcal{V}_A - \dim \mathcal{V}_B$$

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⇝ *Uniqueness of solution for any data* h, g, f $\Leftrightarrow$ $\mathcal{V}_B = \{0\}$

⇝ *In particular, if* $|A| = |B|$ *then*
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Into

Theorem

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- $\rightsquigarrow$ *It has uniqueness of solution for any data* $\Leftrightarrow$ $\mathsf{DN_q(A;B)}$ *has maximum range*
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- From now on, we assume that $\mathsf{DN_q(A; B)}$ is invertible

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can be expressed as $u = \widetilde{\mathcal{G}}_q(h) + \widetilde{\mathcal{N}}_q(g) + \widetilde{\mathcal{P}}_q(f)$ on $\overline{F}$

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Modified Neumann operator

Modified Green and Poisson operators

The unique solution of
$$\begin{cases} \mathcal{L}_q(u) = h & \text{on } F \\ \dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} = g & \text{on } A \\ u = f & \text{on } A \cup R \end{cases}$$

can be expressed as $u = \widetilde{\mathcal{G}}_q(h) + \widetilde{\mathcal{N}}_q(g) + \widetilde{\mathcal{P}}_q(f)$ on $\overline{F}$, where

$$\begin{cases} \mathcal{L}_q\Big(\widetilde{\mathcal{G}}_q(h)\Big) = h \\ \dfrac{\partial \widetilde{\mathcal{G}}_q(h)}{\partial \mathsf{n}_{\mathsf{F}}} = 0 \\ \widetilde{\mathcal{G}}_q(h) = 0 \end{cases} \qquad \begin{cases} \mathcal{L}_q\Big(\widetilde{\mathcal{N}}_q(g)\Big) = 0 \\ \dfrac{\partial \widetilde{\mathcal{N}}_q(g)}{\partial \mathsf{n}_{\mathsf{F}}} = g \\ \widetilde{\mathcal{N}}_q(g) = 0 \end{cases} \qquad \begin{cases} \mathcal{L}_q\Big(\widetilde{\mathcal{P}}_q(h)\Big) = 0 & \text{on } F \\ \dfrac{\partial \widetilde{\mathcal{P}}_q(h)}{\partial \mathsf{n}_{\mathsf{F}}} = 0 & \text{on } A \\ \widetilde{\mathcal{P}}_q(h) = f & \text{on } A \cup R \end{cases}$$

Modified Green operator

Modified Neumann operator

Modified Poisson operator

Modified Green and Poisson operators

$\rightsquigarrow$ We express these modified operators in terms of the classical ones and the matrix $\mathsf{DN}_{\mathsf{q}}(A;B)$

Modified Green and Poisson operators

$\rightsquigarrow$ We express these modified operators in terms of the classical ones and the matrix $\mathsf{DN_q}(\mathsf{A};\mathsf{B})$

Remark

We can not express them in operator terms, as we need to invert a matrix. However, we can do it in matricial terms

Modified Green and Poisson operators

Theorem

$$\begin{aligned}\widetilde{\mathsf{G}_q}(F;F) &= \mathsf{G}_q(F;F) - \mathsf{P}_q(F;B)\cdot \mathsf{DN}_q(A;B)^{-1}\cdot \mathsf{L}_q(A;F)\cdot \mathsf{G}_q(F;F)\\ \widetilde{\mathsf{G}_q}(A\cup R;F) &= 0\\ \widetilde{\mathsf{G}_q}(B;F) &= -\mathsf{DN}_q(A;B)^{-1}\cdot \mathsf{L}_q(A;F)\cdot \mathsf{G}_q(F;F)\end{aligned}$$

Modified Green and Poisson operators

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\widetilde{\mathsf{G}_q}(A\cup R;F) &= 0\\
\widetilde{\mathsf{G}_q}(B;F) &= -\mathsf{DN}_q(A;B)^{-1}\cdot \mathsf{L}_q(A;F)\cdot \mathsf{G}_q(F;F)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathsf{N}_q}(F;A) &= \mathsf{P}_q(F;B)\cdot \mathsf{DN}_q(A;B)^{-1}\\
\widetilde{\mathsf{N}_q}(A\cup R;A) &= 0\\
\widetilde{\mathsf{N}_q}(B;A) &= \mathsf{DN}_q(A;B)^{-1}
\end{aligned}$$

Modified Green and Poisson operators

Theorem

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\widetilde{\mathsf{P}}_{\mathsf{q}}(F;A\cup R) &= \mathsf{P}_{\mathsf{q}}(F;A\cup R) - \mathsf{P}_{\mathsf{q}}(F;B)\cdot \mathsf{DN}_{\mathsf{q}}(A;B)^{-1}\cdot \mathsf{DN}_{\mathsf{q}}(A;A\cup R)\\
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Modified Green and Poisson operators

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\end{aligned}
$$

$\rightsquigarrow$ They can be expressed in terms of the *classical* Green and Poisson operators and of the Dirichlet-to-Neumann map

Partial inverse boundary value problems on finite networks

Partial inverse BVPs on finite networks

$\rightsquigarrow$ We want to obtain the conductances by solving partial BVPs

Partial inverse BVPs on finite networks

- We want to obtain the conductances by solving partial BVPs
- We assume the network is in an equilibrium state

$$\begin{cases} \mathcal{L}_q(u) = 0 & \text{on } F \\ \dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} = g & \text{on } A \\ u = f & \text{on } A \cup R \end{cases}$$

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⇝ We also assume the Dirichlet-to-Neumann map Λ_q to be known, as it can be phisically obtained from electrical boundary measurements

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Remark (Alessandrini 1998, Mandache 2001)

This problem is severelly ill-posed!

Partial inverse BVPs on finite networks

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Remember that under our conditions ($|A| = |B|$ and $\mathsf{DN}_{\mathsf{q}}(\mathsf{A};\mathsf{B})$ invertible) this problem has a unique solution

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Corollary

The unique solution is characterized by the equations

$$\mathsf{u}_B = \mathsf{DN}_q(A;B)^{-1} \cdot \mathsf{g} - \mathsf{DN}_q(A;B)^{-1} \cdot \mathsf{DN}_q(A;A \cup R) \cdot \mathsf{f} \quad \textit{on } B$$

$$u(x) = \mathsf{P}_q(x;A \cup R) \cdot \mathsf{f} + \mathsf{P}_q(x;B) \cdot \mathsf{u}_B \quad \textit{for all } x \in F$$

Partial inverse BVPs on finite networks

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$$u(x) = \mathsf{P_q(x; A \cup R) \cdot f + P_q(x; B) \cdot u_B} \quad \textit{for all } x \in F$$

Remark

Althogh u is not determined yet on F,

Partial inverse BVPs on finite networks

$$\begin{cases} \mathcal{L}_q(u) = 0 & \text{on } F \\ \dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} = g & \text{on } A \\ u = f & \text{on } A \cup R \end{cases}$$

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Remark

Althogh u is not determined yet on F,

$\mathsf{u_B}$ *gives the values of the solution on B!*

Partial inverse BVPs on finite networks

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with all these last steps we only get to know u on $\delta(F)$ and no conductances

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planar network $\Leftrightarrow$ it can be drawn on the plane without crossings between edges

Partial inverse BVPs on finite networks

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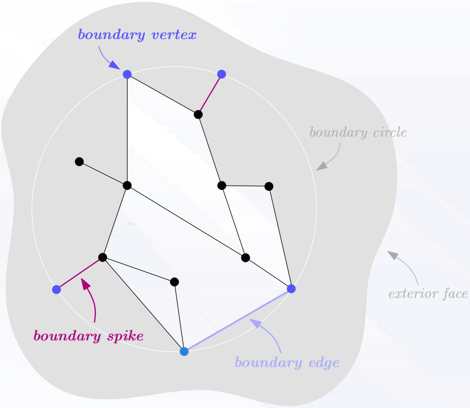
  with all these last steps we only get to know u on $\delta(F)$ and no conductances

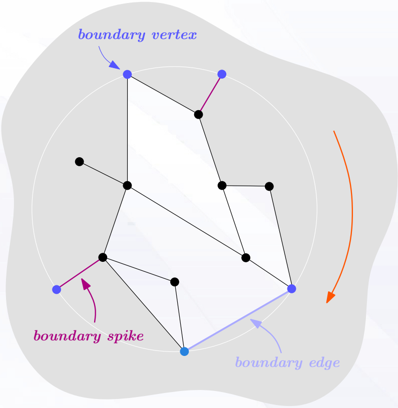
- We restrict to circular planar networks to obtain some conductances

planar network	$\Leftrightarrow$	it can be drawn on the plane without crossings between edges
circular planar network	$\Leftrightarrow$	planar & all the boundary vertices can be found in the same (exterior) face

Partial inverse BVPs on finite networks

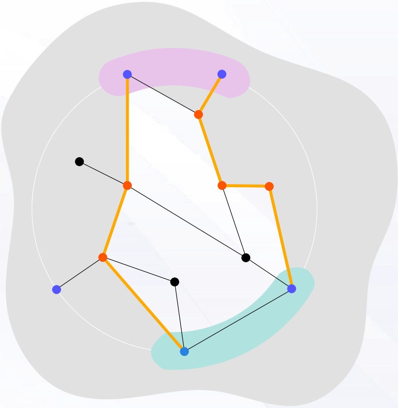


Partial inverse BVPs on finite networks



we will consider certain circular order on the boundary

Partial inverse BVPs on finite networks



a circular pair is connected through the network if there exists a set of disjoint paths between them

Partial inverse BVPs on finite networks

$\rightsquigarrow$ Generalization of Curtis and Morrow's results in 2000

Theorem

(P, Q) circular pair -of size k- of $\delta(F)$, where P and Q are disjoint arcs of the boundary circle

- *(P, Q) not connected through $\Gamma \Leftrightarrow \det\left(\mathsf{DN_q}(\mathsf{P};\mathsf{Q})\right) = 0$.*
- *(P, Q) connected through $\Gamma \Leftrightarrow (-1)^k \det\left(\mathsf{DN_q}(\mathsf{P};\mathsf{Q})\right) > 0$.*

Partial inverse BVPs on finite networks

Corollary (Boundary Spike formula)

If xy is a boundary spike with $y \in \delta(F)$ and contracting xy to a single boundary vertex means breaking the connection through Γ between a circular pair (P, Q), then

$$c(x,y) = \frac{\omega(y)}{\omega(x)} \Big(\mathsf{DN_q}(\mathsf{y};\mathsf{y}) - \mathsf{DN_q}(\mathsf{y};\mathsf{Q}) \cdot \mathsf{DN_q}(\mathsf{P};\mathsf{Q})^{-1} \cdot \mathsf{DN_q}(\mathsf{P};\mathsf{y}) - \lambda \Big)$$

Partial inverse BVPs on finite networks

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Partial inverse BVPs on finite networks

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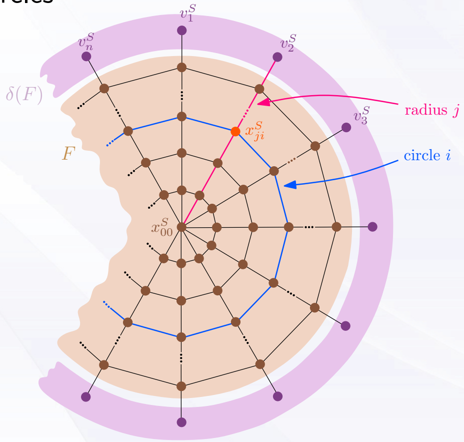
- $\rightsquigarrow$ We can recover certain conductances on planar networks!
- $\rightsquigarrow$ We can try to recover *all* the conductances in special cases: well-connected spider networks

Conductance reconstruction on well-connected spider networks

Conductance reconstruction on w-c spider networks

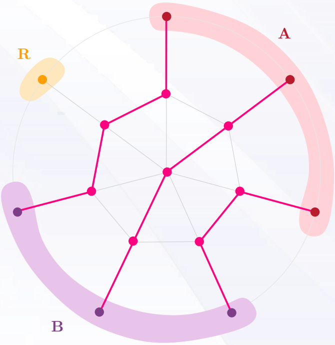
⇝ A well-connected spider network has $n \equiv 3 (\text{mod } 4)$ boundary nodes and $m = \dfrac{n-3}{4}$ circles

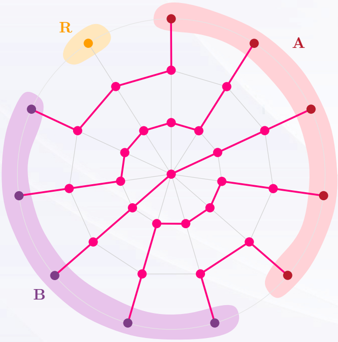


Conductance reconstruction on w-c spider networks

Remark

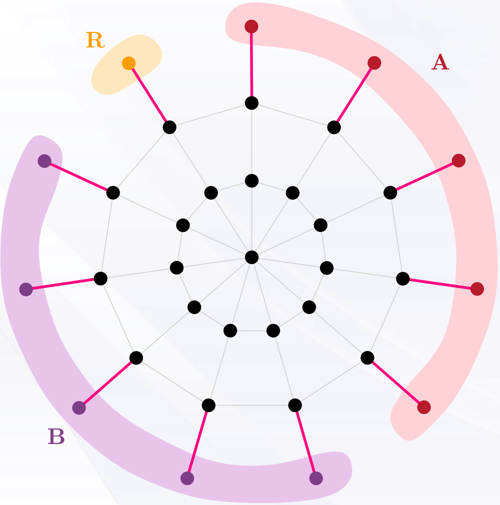
Taking $A = \{v_1^S, \dots, v_{\frac{n-1}{2}}^S\}$, $B = \{v_{\frac{n+1}{2}}^S, \dots, v_{n-1}^S\}$ *and* $R = \{v_n^S\}$ *(or equivalent configurations), then* A *and* B *is a circular pair always connected through the network*





Reconstruction - Step 1

⇝ Boundary spike formula



Reconstruction - Step 2

⇝ We choose $f = \varepsilon_{v_n^S}$ and $g = 0$

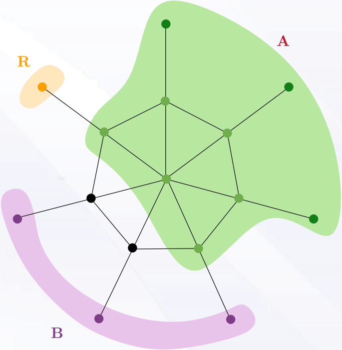
Reconstruction - Step 2

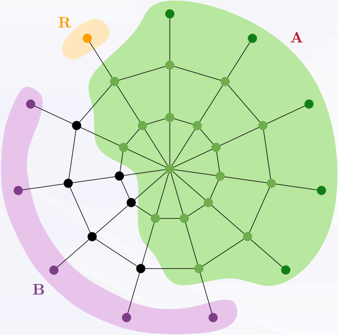
- We choose $f = \varepsilon_{v_n^S}$ and $g = 0$
- Considering problem $\begin{cases} \mathcal{L}_{q_S}(u) = 0 & \text{on } F_S \\ \dfrac{\partial u}{\partial \mathsf{n}_{F_S}} = u = 0 & \text{on } A \\ u = 1 & \text{on } R = \{v_n^S\}, \end{cases}$ then

$$\mathsf{u_B} = -\mathsf{DN_{q_S}}(\mathsf{A};\mathsf{B})^{-1} \cdot \mathsf{DN_{q_S}}(\mathsf{A};\mathsf{v_n^S})$$

Reconstruction - Step 3

⇝ Moreover, we obtain a zero zone of the solution of this BVP problem





Reclaim

$\rightsquigarrow$ We also get to know the values of u on the neighbours of B

Reconstruction - Step 4

$\rightsquigarrow$ We also get to know the values of u on the neighbours of B

$$\mathsf{u}_{\mathsf{N}(\mathsf{B})} = \mathsf{u}_{\mathsf{B}} - \mathsf{L}_{\mathsf{q}_{\mathsf{S}}}(\mathsf{B};\mathsf{N}(\mathsf{B}))^{-1} \cdot \left(\mathsf{DN}_{\mathsf{q}_{\mathsf{S}}}(\mathsf{B};\mathsf{v}_{\mathsf{n}}^{\mathsf{S}}) + \mathsf{DN}_{\mathsf{q}_{\mathsf{S}}}(\mathsf{B};\mathsf{B}) \cdot \mathsf{u}_{\mathsf{B}}\right)$$

Reconstruction - Step 4

⇝ We also get to know the values of u on the neighbours of B

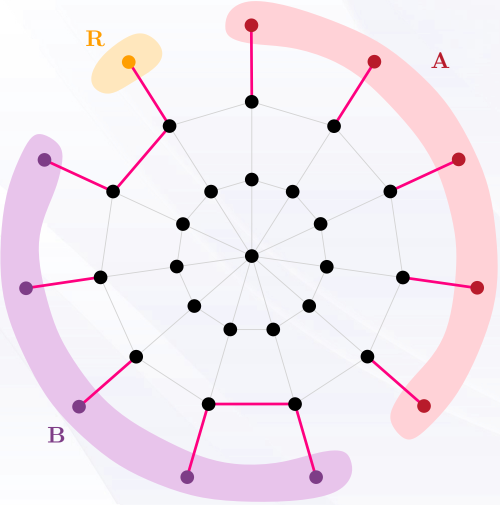
$$u_{N(B)} = u_B - L_{q_S}(B; N(B))^{-1} \cdot \left(DN_{q_S}(B; v_n^S) + DN_{q_S}(B; B) \cdot u_B \right)$$

↑ ↑ ↑ ↑ ↑

already known!

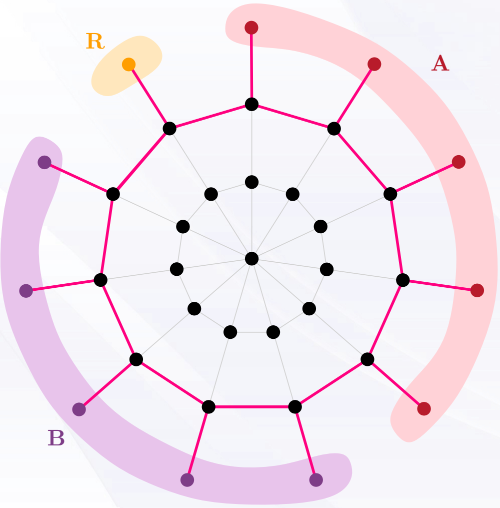
Reconstruction - Step 5

⇝ With this information, we obtain two new conductances



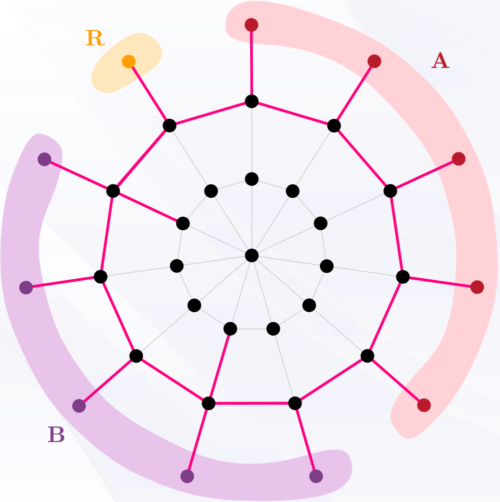
Reconstruction - Step 6

⇝ ...and rotating the BVP, we obtain more conductances



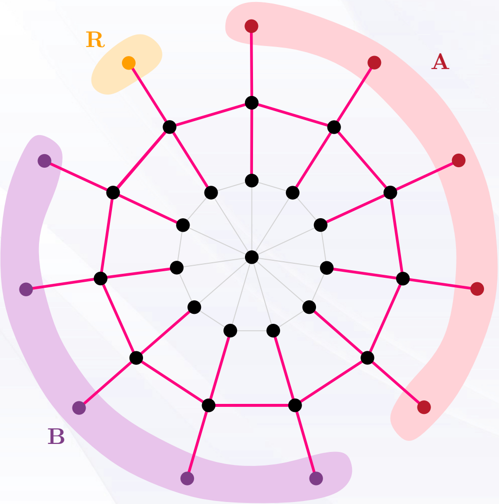
Reconstruction - Step 7

⇝ Now we can even obtain two more conductances



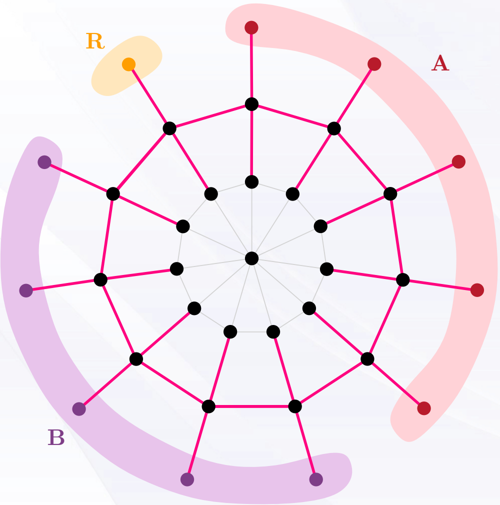
Reconstruction - Step 8

⇝ ...and rotating the BVP again,



Reconstruction - Step 9 and forward

⇝ Working analogously, we finally get all the conductances



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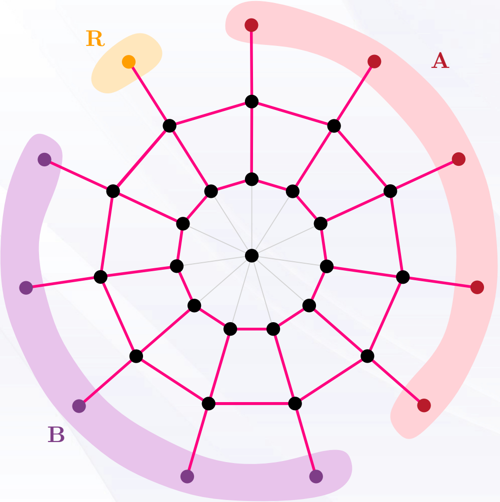

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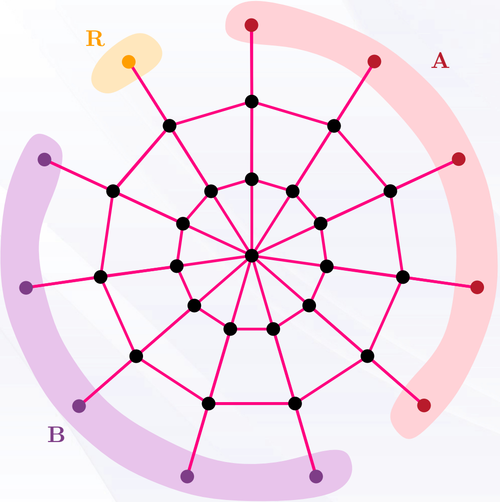

Gràcies!