# On non-reducible quasi-periodic linear skew-products

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# Recall: Floquet theory I

Floquet theory named after Gaston Floquet (S.XIX). Concerns about linear equations with periodic coefficients.

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad A(t+T) = A(t).$$

Where $A$ depends continuously on time. Denote it by $T$-LSPC.

# Recall: Floquet theory II

Floquet theorem

- States that every $T$-LSPC can be reduced to a linear system with constant coefficients.
- It is done by means of a $T$-periodic change of variables which may be complex.
- There always exists a $2T$-periodic real change of variables reducing the system.

# Two dimensional quasi-periodic linear skew-products

A discrete system

$$\begin{cases} \bar{x} = A(\theta)x, \\ \bar{\theta} = \theta + \omega, \end{cases}$$

where $x \in \mathbb{R}^2$; $\theta \in \mathbb{T}$; $A \in C^0(\mathbb{T}, \mathrm{GL}_2\,\mathbb{R})$ and $\omega \notin 2\pi\mathbb{Q}/[0, 2\pi]$, is called a q.p linear skew product (QPLSP).

**Importance**: Discrete systems has they own interest and, moreover, can be used to study the q.p linear differential equations.

# Reducibility of QPLSP

## Definition

A QPLSP is said to be reducible if there exists a continuous change of variables $x = C(\theta)y$ such that transforms the former system to:

$$\begin{cases} \bar{y} = By \\ \bar{\theta} = \theta + \omega \end{cases}$$

Where $B = C^{-1}(\theta + \omega)A(\theta)C(\theta)$ does not depend on $\theta$.

# Winding number of a matrix

**Definition**

Let $\theta \in \mathbb{T}$, $A \in C^0(\mathbb{T}, \mathrm{GL}_2\,\mathbb{R})$. Fix a vector $v \in \mathbb{R}^2$; $v \neq 0$, and consider the planar curve $v_A(\theta) = A(\theta)v$, which does not pass through the origin. We define the winding number of the matrix $A(\theta)$ as the winding number of $v_A(\theta)$ around the origin.

**Lemma**

*The winding number of a matrix, defined as above, does not depend on the choice of the vector $v$.*

# The winding number of a product

## Theorem

*Let* $A, B \in C^0(\mathbb{T}, \mathrm{GL}_2\,\mathbb{R})$*, then*

$$\mathrm{wind}(A(\theta)B(\theta)) = \mathrm{wind}(A(\theta)) + \mathrm{wind}(B(\theta)).$$

*Where* wind *stands for the winding number.*

## Corollary

*The winding number of a matrix* $A(\theta)$ *is invariant under changes of the type* $C^{-1}(\theta + \omega)A(\theta)C(\theta)$*.*

# Detecting non-reducibility

### Criterion

Given a planar QPLSP

$$\begin{cases} \bar{x} = A(\theta)x \\ \bar{\theta} = \theta + \omega \end{cases}$$

If $\mathrm{wind}(A(\theta)) \neq 0$, the system cannot be reducible.

# Examples

The rotation matrices

$$R_\ell(\theta) = \begin{pmatrix} \cos \ell\theta & -\sin \ell\theta \\ \sin \ell\theta & \cos \ell\theta \end{pmatrix}.$$

have wind $R_\ell(\theta) = \ell$ and hence they cannot be reducible.

**Remark**

The winding number of a Poincaré map associated to a planar quasi-periodic linear differential equation is zero.

# Reducibility and dynamics

**Question**

Does reducibility manifests in dynamics?

**Question**

How we can see its impact?

# Affine Systems

Consider the following affine system.

$$\begin{cases} \bar{x} = \mu A(\theta)x + b(\theta) \\ \bar{\theta} = \theta + \omega, \end{cases}$$

Where $\theta \in \mathbb{T}$, $\mu > 0$ is a parameter, $b(\theta) \in C^0(\mathbb{T}, \mathbb{R}^2) =: E$ and $A \in C^0(\mathbb{T}, \mathrm{GL}_2 \mathbb{R})$. Henceforth we shall write $\|A\|$, meaning the norm of the matrix $A$ as a linear operator of $E$.

**Question**

Do AF have invariant curves?

# Existence and uniqueness of invariant curves

**Theorem**

*For each $\mu$, the AS introduced before has exactly one invariant curve $x^*$ whenever $\mu \notin [\|A\|^{-1}, \|A^{-1}\|]$. Set $k_T = \mu\|A\|$ and $k_P = \mu^{-1}\|A^{-1}\|$.*

- *If $\mu < \|A\|^{-1}$ then $x^*$ is AS and $\|x^*\| < \frac{1}{1-k_T}\|b\|$.*
- *If $\mu > \|A^{-1}\|$ then $x^*$ is AU ans $\|x^*\| < \frac{k_P}{1-k_P}\|b\|$.*

**Remark**

Proof: Apply the Fixed Point Theorem for Banach spaces.

# Exploring concrete cases

We study the following AS in $\mathbb{C}$.

$$\begin{cases} \bar{z} = \mu e^{i\theta} z + 1, \\ \bar{\theta} = \theta + \omega. \end{cases}$$

We already know there is an invariant curve if $\mu \neq 1$. Notice that $\Lambda(x_{0)} = \Lambda = \ln \mu$. There is no invariant curve for $\mu = 1$. In the reducible case, the invariant curve collapses to a point:

$$x^* = \frac{1}{1 - \mu\alpha},$$

where $\alpha$ is the coefficient of the reduced system.

# Explicit invariant curves

**Theorem**

*The solutions of the invariant curve equation are given by:*

$$z(\theta) = \sum_{k=0}^{\infty} \mu^k e^{-i\frac{k(k+1)}{2}\omega} e^{ik\theta}, \quad \mu < 1$$

*and*

$$z(\theta) = -\sum_{k=0}^{-\infty} \mu^k e^{-i\frac{k(k+1)}{2}\omega} e^{ik\theta}, \quad \mu > 1$$

From now on, we will work only on the case $\mu < 1$.

# Numerical Experiments

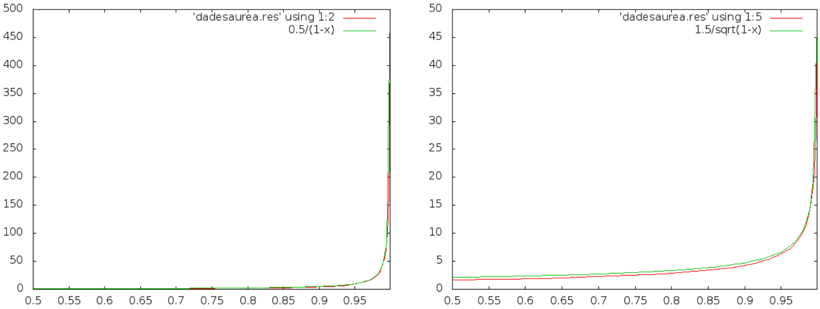


Figure: Fittings of the winding number and the infinity norm.

# Numerical Experiments II

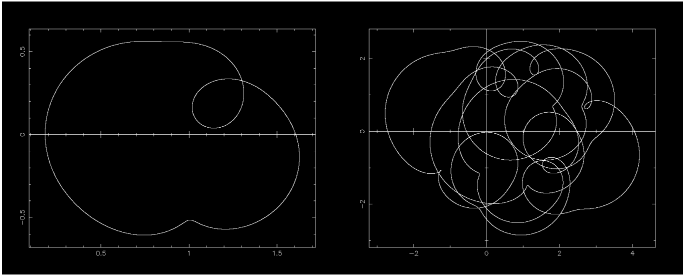


Figure: Attractors for $\mu = 0.5$ and $\mu = 0.9$.

# Numerical Experiments III

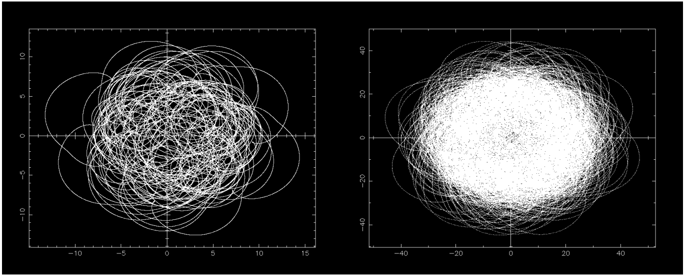


Figure: Attractors for $\mu = 0.99$ and $\mu = 0.999$.

# On the infinity norm

Name:

$$z_\mu(\theta) = \sum_{k=0}^{\infty} \mu^k C_k^\omega e^{ik\theta}, \qquad C_k^\omega = e^{-i\frac{k(k+1)}{2}\omega}.$$

**Fact**

*The growth of the infinity norm:*

$$\frac{1}{\sqrt{1-\mu}} \leq \|z_\mu\|_\infty \leq \frac{1}{1-\mu}.$$

*If the sequence* $\{C_k^\omega\}_{k\in\mathbb{Z}}$ *is equidistributed on the circle, then:*

$$\|z_\mu\|_\infty \sim \frac{1}{\sqrt{1-\mu}}.$$

# On the winding number

**Fact**

*If $\{C_k^\omega\}_{k\in\mathbb{Z}}$ is equidistributed on the circle, the winding number of $z_\mu$ gets arbitrarily large when $\mu$ gets close to 1.*

**Remark**

To prove the last fact is equivalent to prove that the function

$$f(\rho) = \sum_{k=0}^{\infty} C_k^\omega \rho^k$$

has infinitely many zeros in the disk of radius 1.

# WOW


Figure: WOW, such SIMBa.