Stochastic Differential Equations

SIMBA, Barcelona.
David Baños

UiO : Faculty of Mathematics and Natural Sciences
University of Oslo

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Ordinary Differential Equation

Let $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $x : [0, T] \to \mathbb{R}^d$

$$dx(t) = f(t, x(t))dt, \quad t_0 \leqslant t \leqslant T, \quad x(t_0) = x_0 \in \mathbb{R}^d. \tag{1}$$

If a solution to the Cauchy problem (1) exists we can write

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \quad t_0 \leqslant t \leqslant T. \tag{2}$$

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Example

If $f(s, x(s)) = x(s)$ then (1) has a closed form solution given by

$$x(t) = x_0 e^t.$$

Existence and Uniqueness of Solutions

- Picard-Lindelöf theorem (suff. cond. for local existence and uniqueness)
- Peano's theorem (suff. cond. for existence)
- Carathéodory's theorem (weaker version of Peano's theorem)
- Okamura's theorem (nec. an suff. conditions for uniqueness)
- ...

Brownian motion

In 1827, while examining **grains of pollen** of the plant Clarkia pulchella **suspended in water** under a microscope, Brown observed **small particles ejected from the pollen grains**, executing a **continuous jittery motion**. He then observed the same motion in particles of inorganic matter, enabling him to rule out the hypothesis that the effect was life-related. Although Brown did not provide a theory to explain the motion, and Jan Ingenhousz already had reported a similar effect using charcoal particles, in German and French publications of 1784 and 1785, the phenomenon is now known as **Brownian motion**.

Brownian motion and SDE's

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process $B : [0, T] \times \Omega \to \mathbb{R}^d$ defined on $(\Omega, \mathcal{F}, P)$ is said to be a **standard Brownian motion** (moviment Brownià) or a **Wiener process** if

1. $B_0 = 0, \; P - a.s.$
2. Given two times $s, t \in [0, T]$, $s < t$, the law of $B_{t+s} - B_s$ is the same as B_t.
3. The increments $B_t - B_s$ and $B_v - B_u$ are independent for all $u < v$, $s < t$.
4. $B_t \sim N(0, t)$ for all $t \in [0, T]$.

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4. $B_t \sim N(0, t)$ for all $t \in [0, T]$.

In addition, we can choose a *version* such that B_t is almost surely continuous. The existence of a stochastic process defined as above is not immediate (Kolmogorov's existence theorem).

Brownian motion

To have intuition working with $\{B_t(\omega), t \in [0, T], \omega \in \Omega\}$ we present four *sample paths* of a standard Brownian motion.

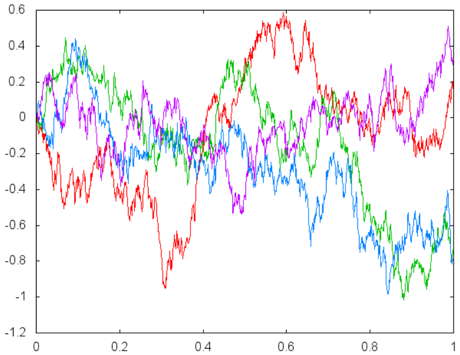

FIGURE: Four realizations of a *standard Brownian motion* on the interval $[0, 1]$.

Stochastic Differential Equation

A (ordinary) **stochastic differential equation** with **additive noise** is an equation of the form:

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma dB_t, \quad t \in [0, T], \\ X_0 &= x \in \mathbb{R}^d \end{aligned} \tag{3}$$

where σ is a parameter often called **volatility**.

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Observe that for each $\omega \in \Omega$

$$X_t(\omega) = x + \int_0^t b(s, X_s(\omega))ds + \sigma B_t(\omega).$$

Stochastic Differential Equation

A (ordinary) **stochastic differential equation** is an equation of the form:

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \in [0, T], \\ X_0 &= x \in \mathbb{R}^d \end{aligned} \tag{4}$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ is a suitable function.

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Stochastic Differential Equation

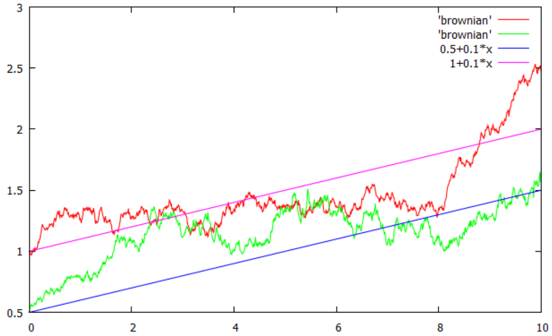

FIGURE: Two samples of Brownian motion with drift at different starting points.

Stochastic Differential Equation

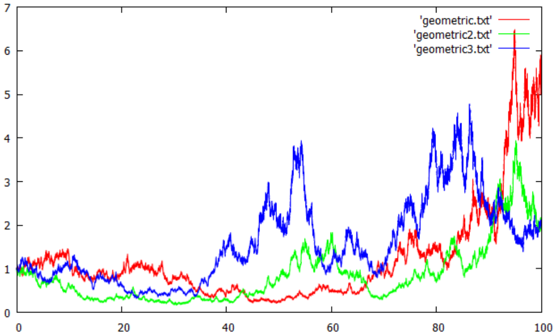

FIGURE: Three samples fo a geometric Brownian motion with $\mu = 0.05$ and $\sigma = 0.02$.

Path properties of the Brownian motion

The function the function $t \mapsto B_t(\omega)$ has the following properties:

- Takes both strictly positive and strictly negative numbers on $(0, \varepsilon)$ for every $\varepsilon > 0$.
- It is continuous everywhere but differentiable nowhere.
- It has infinite variation.
- Finite quadratic variation.
- The set of zeros is a nowhere dense perfect set of Lebesgue measure 0 and Hausdorff dimension $1/2$.
- Hölder-continuous paths of index $\alpha < 1/2$.
- Hausdorff dimension 1.5.

Stochastic integral

Let $B_t(\omega)$, $t \in [0, T]$, $\omega \in \Omega$ be a standard Brownian motion. Consider a process X_t satisfying some conditions. Then

$$\int_0^T X_t dB_t := \lim_{n\to\infty} \sum_{[t_i,t_{i+1}]\in\pi_n} X_{t_i}(B_{t_{i+1}} - B_{t_i}) \quad \text{(Itô integral)}$$

$$\int_0^T X_t dB_t := \lim_{n\to\infty} \sum_{[t_i,t_{i+1}]\in\pi_n} X_{\frac{t_i+t_{i+1}}{2}}(B_{t_{i+1}} - B_{t_i}) \quad \text{(Stratonovich integral)}$$

The convergence above is in probability (in fact, in $L^2(\Omega)$).

Back to ODE theory

Consider the Cauchy problem

$$\begin{cases} dx(t) = b(t, x(t))dt, \ \ t \in [0, T], \\ x(t_0) = x_0 \in \mathbb{R}. \end{cases}$$

Theorem (Picard-Lindelöf)

If b is continuous in t and Lipschitz continuous in x then there exists a unique (local) strong solution to the Cauchy problem above.

Back to SDE theory

Consider the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + dB_t, \ \ t \in [0, T], \\ X_0 = x \in \mathbb{R}. \end{cases}$$

Theorem (Stochastic version of Existence and Uniqueness)

If b satisfies one of the following conditions then there exists a unique (global) strong solution to SDE above.

- *b is Lipschitz continuous in x uniformly in t.*
- *b is bounded and measurable.*
- *b is of linear growth, i.e.* $|b(t, x)| \leqslant C(1 + |x|)$.
- *b satisfies* $\int_0^T \left(\int_{\mathbb{R}} |b(t, x)|^q dx\right)^{p/q} dt < \infty$.
- ...

Densities of solutions to SDE's

X_t is a process which for each fixed t, X_t is a random variable and hence it has a law but not necessarily a density.

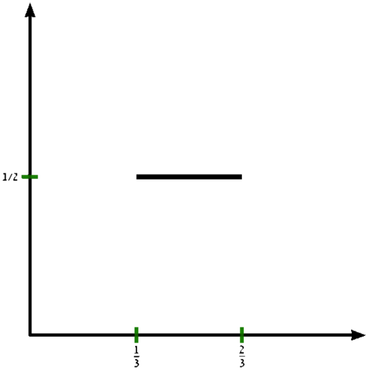

Figure: Devil's staircase function. (taken from Wikipedia)

Densities of solutions to SDE's

Hence, it is not even clear whether the solution to an SDE has a density! A sufficient condition for X_t to admit a density is the following:

$$E\left[\exp\left\{\frac{1}{2}\int_0^T b(X_t)^2 dt\right\}\right] < \infty \quad \text{(Novikov's condition)}$$

Densities of solutions to SDE's

A lot of research on this direction has been done. Given an SDE with some conditions on b and σ. Does X_t admit a density? How regular is it?

References

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Thank you!