# A Note on Mixed Norm Spaces

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# Rearrangement-invariant Banach function spaces

Given a finite interval $\mathtt{I}$, we denote $\mathtt{I}^n = \overbrace{\mathtt{I} \times \ldots \times \mathtt{I}}^{n}$, $n \in \mathbb{N}$.

**Definition**

A rearrangement invariant Banach function space (briefly an r.i. space) is defined as

$$X(\mathtt{I}^n) = \left\{ f \in \mathcal{M}(\mathtt{I}^n) : \left\| f \right\|_{X(\mathtt{I}^n)} < \infty \right\},$$

where $\left\| \cdot \right\|_{X(\mathtt{I}^n)}$ satisfies certain properties.

**Examples** The Lebesgue spaces $L^p(\mathtt{I}^n)$, where

$$\left\| f \right\|_{L^p(\mathtt{I}^n)} = \begin{cases} \left( \int_{\mathtt{I}^n} |f(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty; \\ \inf \left\{ C \geq 0 : |f(x)| \leq C \text{ a.e} \right\}, & p = \infty. \end{cases}$$

# Mixed norm spaces

Let $n \in \mathbb{N}$, $n \geq 2$ and $k \in \{1, \ldots, n\}$. For any $x \in \mathtt{I}^n$, we denote

$$\widehat{x_k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathtt{I}^{n-1}.$$

**Definition**

The Benedek-Panzone spaces are defined as

$$\mathcal{R}_k(X, Y) = \left\{ f \in \mathcal{M}(\mathtt{I}^n) : \left\| f \right\|_{\mathcal{R}_k(X,Y)} < \infty \right\},$$

where $\left\| f \right\|_{\mathcal{R}_k(X,Y)} = \left\| \psi_k(f, Y) \right\|_{X(\mathtt{I}^{n-1})}, \quad \psi_k(f, Y)(\widehat{x_k}) = \left\| f(\widehat{x_k}, \cdot) \right\|_{Y(\mathtt{I})}.$

# Mixed norm spaces

**Examples:**

The Lebesgue spaces $L^1(\mathtt{I}) = \mathcal{R}_1(L^1, L^1)$;

$$\|f\|_{\mathcal{R}_1(L^1,L^1)} = \int_{\mathtt{I}^{n-1}} \psi_1(f, L^1)(\widehat{x_1}) d\widehat{x_1} = \int_{\mathtt{I}^{n-1}} \int_{\mathtt{I}} |f(\widehat{x_1}, x_1)| dx_1 \, d\widehat{x_1}.$$

The Benedek-Panzone spaces $\mathcal{R}_n(L^1, L^2)$;

$$\|f\|_{\mathcal{R}_n(L^1,L^2)} = \int_{\mathtt{I}^{n-1}} \psi_n(f, L^2)(\widehat{x_n}) d\widehat{x_n} = \int_{\mathtt{I}^{n-1}} \left( \int_{\mathtt{I}} |f(\widehat{x_n}, x_n)|^2 dx_n \right)^{1/2} d\widehat{x_n}.$$

The Benedek-Panzone spaces $\mathcal{R}_k(L^3, L^\infty)$;

$$\|f\|_{\mathcal{R}_k(L^3,L^\infty)} = \left( \int_{\mathtt{I}^{n-1}} \left[\psi_k(f, L^\infty)(\widehat{x_k})\right]^3 d\widehat{x_k} \right)^{1/3} = \left( \int_{\mathtt{I}^{n-1}} \left\|f(\widehat{x_k}, \cdot)\right\|_{L^\infty(I)}^3 d\widehat{x_k} \right)^{1/3}.$$

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# Mixed norm spaces

**Definition**

The mixed norm spaces $\mathcal{R}(X, Y)$ are defined as follows

$$\mathcal{R}(X, Y) = \bigcap_{k=1}^{n} \mathcal{R}_k(X, Y).$$

For each $f \in \mathcal{R}(X, Y)$, we set $\|f\|_{\mathcal{R}(X,Y)} = \sum_{k=1}^{n} \|f\|_{\mathcal{R}_k(X,Y)}$.

**Examples:**

The Lebesgue spaces $L^p(\mathrm{I}^n) = \mathcal{R}(L^p, L^p)$, $1 \leq p \leq \infty$.

# Sobolev spaces

We denote $\nabla u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$, where $\partial_{x_i} u$ is the distributional partial derivate of $u$ with respect to $x_i$.

**Definition**

The first-order Sobolev spaces are defined as

$$W^1 Z(\mathtt{I}^n) := \left\{ u \in L^1_{\mathrm{loc}}(\mathtt{I}^n) : u \in Z(\mathtt{I}^n) \text{ and } |\nabla u| \in Z(\mathtt{I}^n) \right\},$$

with the norm $\|u\|_{W^1 Z(\mathtt{I}^n)} = \|u\|_{Z(\mathtt{I}^n)} + \||\nabla u|\|_{Z(\mathtt{I}^n)}$.

By $W^1_0 Z(\mathtt{I}^n)$ we denote the closure of $C^\infty_c(\mathtt{I}^n)$ in $W^1 Z(\mathtt{I}^n)$.

Classical Sobolev embedding theorem
$W_0^1 L^p(\mathtt{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathtt{I}^n), \quad 1 \leq p < n.$

Sobolev, case $p > 1$.
His proof did not apply to p = 1.

Gagliardo; Nirenberg, $p = 1$.
$W_0^1 L^1(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n'}(\mathtt{I}^n).$

Fournier embedding theorem
$\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(\mathtt{I}^n).$

$$W_0^1 L^1(\mathtt{I}^n) \hookrightarrow L^{n',1}(\mathtt{I}^n) \hookrightarrow L^{n'}(\mathtt{I}^n).$$

# Sobolev embeddings in r.i. spaces

Kerman and Pick studied the Sobolev embeddings among r.i. spaces. In particular, they solved the following problems:

* Given an r.i. range space $X(\mathtt{I}^n)$, find the largest r.i. domain space, with a.c. norm, namely $Z(\mathtt{I}^n)$, satisfying

  $$W_0^1 Z(\mathtt{I}^n) \hookrightarrow X(\mathtt{I}^n).$$

  This means that if $W_0^1 \widetilde{Z}(\mathtt{I}^n) \hookrightarrow X(\mathtt{I}^n) \Rightarrow \widetilde{Z}(\mathtt{I}^n) \hookrightarrow Z(\mathtt{I}^n)$.

* Given an r.i. domain space $Z(\mathtt{I}^n)$, describe the smallest r.i. range space, namely $X(\mathtt{I}^n)$, that verifies

  $$W_0^1 Z(\mathtt{I}^n) \hookrightarrow X(\mathtt{I}^n).$$

  That is, if $W_0^1 Z(\mathtt{I}^n) \hookrightarrow \widetilde{X}(\mathtt{I}^n) \Rightarrow X(\mathtt{I}^n) \hookrightarrow \widetilde{X}(\mathtt{I}^n)$.

# Examples

Classical Sobolev embedding theorem

$W_0^1 L^p(\mathtt{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathtt{I}^n), \quad 1 \le p < n.$

↓

Hunt; O'Neil; Peetre.

$W_0^1 L^p(\mathtt{I}^n) \hookrightarrow L^{pn/(n-p),p}(\mathtt{I}^n).$

↙ ↘

Kerman and Pick

Optimal r.i. domain space: $L^p(\mathtt{I}^n)$.

Kerman and Pick

Optimal r.i. range space: $L^{pn/(n-p),p}(\mathtt{I}^n)$.

# Examples

Critical Sobolev embedding theorem

$W_0^1 L^n(I^n) \hookrightarrow L^p(I^n)$, $1 \leq p < \infty$.

Maz'ya; Hansson; Brézis and Wainger

$W_0^1 L^n(I^n) \hookrightarrow L^{\infty,n;-1}(I^n)$.

Kerman and Pick

Optimal r.i. domain space: $Z_{L^{\infty,n;-1}}(I^n)$.

Hansson; Kerman and Pick

Optimal r.i. range space: $L^{\infty,n;-1}(I^n)$.

# Motivation

Classical Sobolev embeddings

Gagliardo; Nirenberg

$W_0^1 L^1(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^1, L^\infty).$

Kerman and Pick

Optimal domain and range of

$W_0^1 Z(\mathtt{I}^n) \hookrightarrow Y(\mathtt{I}^n),$

within the class of r.i. spaces.

Describe the largest domain space and the smallest range with mixed norm in $W_0^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(X, L^\infty)$.

# Problem

Let $X(I^{n-1})$ be an r.i. space and let $Z(\mathrm{I}^n)$ be an r.i space, with a.c. norm. Our aim is to study the Sobolev embedding

$$W_0^1 Z(\mathrm{I}^n) \hookrightarrow \mathcal{R}(X, L^\infty). \tag{1}$$

We are interested in the following questions:

- Given a mixed norm space $\mathcal{R}(X, L^\infty)$, we want to find the largest r.i. domain space, with a.c. norm, satisfying (1).
- Let $Z(\mathrm{I}^n)$ be an r.i. domain space, with a.c. norm. We would like to find the smallest range space of the form $\mathcal{R}(X, L^\infty)$ for which (1) holds.

# Examples

Classical Sobolev embedding theorem

$$W_0^1 L^p(\mathtt{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathtt{I}^n), \quad 1 \le p < n.$$

↓

$$W_0^1 L^p(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty).$$

↙ ↘

$L^p(\mathtt{I}^n)$ optimal r.i. domain space.

$\mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty)$ optimal range of the form $\mathcal{R}(X, L^\infty)$.

# Examples

Critical Sobolev embedding theorem
$W_0^1 L^n(\mathrm{I}^n) \hookrightarrow L^p(\mathrm{I}^n)$, $1 \leq p < \infty$.

↓

$W_0^1 L^n(\mathrm{I}^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty)$.

↙ ↘

$Z_{L^{\infty,n;-1}}(\mathrm{I}^n)$ optimal r.i. domain space.

$\mathcal{R}(L^{\infty,n;-1}, L^\infty)$ optimal range of the form $\mathcal{R}(X, L^\infty)$.

# Domain space: $L^p(\mathtt{I}^n)$, $1 \le p < n$

Classical Sobolev embedding theorem

$$W_0^1 L^p(\mathtt{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathtt{I}^n), \quad 1 \le p < n.$$

Kerman and Pick

$L^{pn/(n-p),p}(\mathtt{I}^n)$ optimal r.i. range space.

$\mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty)$ optimal range of the form $\mathcal{R}(X, L^\infty)$.

$$W_0^1 L^p(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-p),p}, L^\infty) \underset{\neq}{\hookrightarrow} L^{pn/(n-p)}(\mathtt{I}^n).$$

# Domain space: $L^n(\mathtt{I}^n)$

Critical Sobolev embedding theorem

$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow L^p(\mathtt{I}^n)$, $1 \leq p < \infty$.

Kerman and Pick

$L^{\infty,n;-1}(\mathtt{I}^n)$ optimal r.i. range space.

$\mathcal{R}(L^{\infty,n;-1}, L^\infty)$ optimal range of the form $\mathcal{R}(X, L^\infty)$.

$$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(\mathtt{I}^n).$$

# Domain space: $Z(\mathtt{I}^n)$

$Z(\mathtt{I}^n)$ r.i. domain space.

$X^{\mathrm{op}}(\mathtt{I}^n)$ optimal r.i. range space for $W_0^1 Z(\mathtt{I}^n) \hookrightarrow X^{\mathrm{op}}(\mathtt{I}^n)$.

$\mathcal{R}(Y^{\mathrm{op}}, L^\infty)$ optimal range of the form $\mathcal{R}(X, L^\infty)$ for $W_0^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(Y^{\mathrm{op}}, L^\infty)$.

$W_0^1 Z(\mathtt{I}^n) \hookrightarrow \mathcal{R}(Y^{\mathrm{op}}, L^\infty) \hookrightarrow X^{\mathrm{op}}(\mathtt{I}^n)$.

# Critical case of the classical Sobolev embedding

Critical Sobolev embedding theorem

↓

Hansson; Kerman and Pick

Optimal r.i range space

$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow L^{\infty,n;-1}(\mathtt{I}^n).$

↓

Bastero, Milman and Ruiz; Malý and Pick

Improvement among non-linear r.i. spaces

$W_0^1 L^n(\mathtt{I}^n) \hookrightarrow L(\infty,n)(\mathtt{I}^n) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(\mathtt{I}^n).$

# Critical case of the classical Sobolev embedding

Critical Sobolev embedding theorem

↓

Optimal range of the form $\mathcal{R}(X, L^\infty)$

$$W_0^1 L^n(\mathrm{I}^n) \hookrightarrow L^{\infty,n;-1}(\mathrm{I}^n).$$

↓

Improvement among non-linear spaces of the form $\mathcal{R}(X, L^\infty)$

$$W_0^1 L^n(\mathrm{I}^n) \hookrightarrow \mathcal{R}(L(\infty, n), L^\infty) \underset{\neq}{\hookrightarrow} \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

# The end

**Thank You!!**