

A Note on Mixed Norm Spaces

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- 4 Critical case of the classical Sobolev embedding

Rearrangement-invariant Banach function spaces

Given a finite interval I , we denote $I^n = \overbrace{I \times \dots \times I}^n$, $n \in \mathbb{N}$.

Definition

A **rearrangement invariant Banach function space** (briefly an r.i. space) is defined as

$$X(I^n) = \{f \in \mathcal{M}(I^n) : \|f\|_{X(I^n)} < \infty\},$$

where $\|\cdot\|_{X(I^n)}$ satisfies certain properties.

Examples The Lebesgue spaces $L^p(I^n)$, where

$$\|f\|_{L^p(I^n)} = \begin{cases} \left(\int_{I^n} |f(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty; \\ \inf \{C \geq 0 : |f(x)| \leq C \text{ a.e.}\}, & p = \infty. \end{cases}$$

Mixed norm spaces

Let $n \in \mathbb{N}$, $n \geq 2$ and $k \in \{1, \dots, n\}$. For any $x \in \mathbb{I}^n$, we denote

$$\hat{x}_k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \mathbb{I}^{n-1}.$$

Definition

The **Benedek-Panzone spaces** are defined as

$$\mathcal{R}_k(X, Y) = \{f \in \mathcal{M}(\mathbb{I}^n) : \|f\|_{\mathcal{R}_k(X, Y)} < \infty\},$$

where $\|f\|_{\mathcal{R}_k(X, Y)} = \|\psi_k(f, Y)\|_{X(\mathbb{I}^{n-1})}$, $\psi_k(f, Y)(\hat{x}_k) = \|f(\hat{x}_k, \cdot)\|_{Y(\mathbb{I})}$.

Mixed norm spaces

Examples:

The Lebesgue spaces $L^1(\mathbb{I}) = \mathcal{R}_1(L^1, L^1)$;

$$\|f\|_{\mathcal{R}_1(L^1, L^1)} = \int_{\mathbb{I}^{n-1}} \psi_1(f, L^1)(\hat{x}_1) d\hat{x}_1 = \int_{\mathbb{I}^{n-1}} \int_{\mathbb{I}} |f(\hat{x}_1, x_1)| dx_1 d\hat{x}_1.$$

The Benedek-Panzone spaces $\mathcal{R}_n(L^1, L^2)$;

$$\|f\|_{\mathcal{R}_n(L^1, L^2)} = \int_{\mathbb{I}^{n-1}} \psi_n(f, L^2)(\hat{x}_n) d\hat{x}_n = \int_{\mathbb{I}^{n-1}} \left(\int_{\mathbb{I}} |f(\hat{x}_n, x_n)|^2 dx_n \right)^{1/2} d\hat{x}_n.$$

The Benedek-Panzone spaces $\mathcal{R}_k(L^3, L^\infty)$;

$$\|f\|_{\mathcal{R}_k(L^3, L^\infty)} = \left(\int_{\mathbb{I}^{n-1}} [\psi_k(f, L^\infty)(\hat{x}_k)]^3 d\hat{x}_k \right)^{1/3} = \left(\int_{\mathbb{I}^{n-1}} \|f(\hat{x}_k, \cdot)\|_{L^\infty(\mathbb{I})}^3 d\hat{x}_k \right)^{1/3}.$$

Mixed norm spaces

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Mixed norm spaces

Definition

The mixed norm spaces $\mathcal{R}(X, Y)$ are defined as follows

$$\mathcal{R}(X, Y) = \bigcap_{k=1}^n \mathcal{R}_k(X, Y).$$

For each $f \in \mathcal{R}(X, Y)$, we set $\|f\|_{\mathcal{R}(X, Y)} = \sum_{k=1}^n \|f\|_{\mathcal{R}_k(X, Y)}$.

Examples:

The Lebesgue spaces $L^p(\mathbb{I}^n) = \mathcal{R}(L^p, L^p)$, $1 \leq p \leq \infty$.

Sobolev spaces

We denote $\nabla u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$, where $\partial_{x_i} u$ is the distributional partial derivate of u with respect to x_i .

Definition

The **first-order Sobolev spaces** are defined as

$$W^1 Z(\mathbb{I}^n) := \{u \in L^1_{\text{loc}}(\mathbb{I}^n) : u \in Z(\mathbb{I}^n) \text{ and } |\nabla u| \in Z(\mathbb{I}^n)\},$$

with the norm $\|u\|_{W^1 Z(\mathbb{I}^n)} = \|u\|_{Z(\mathbb{I}^n)} + \|\nabla u\|_{Z(\mathbb{I}^n)}$.

By $W^1_0 Z(\mathbb{I}^n)$ we denote the closure of $C_c^\infty(\mathbb{I}^n)$ in $W^1 Z(\mathbb{I}^n)$.

Classical Sobolev embedding theorem
 $W_0^1 L^p(\mathbb{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathbb{I}^n), \quad 1 \leq p < n.$



Sobolev, case $p > 1$.
 His proof did not apply to $p = 1$.



Gagliardo; Nirenberg, $p = 1$.
 $W_0^1 L^1(\mathbb{I}^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n'}(\mathbb{I}^n).$

Fournier embedding theorem
 $\mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n',1}(\mathbb{I}^n).$



$W_0^1 L^1(\mathbb{I}^n) \hookrightarrow L^{n',1}(\mathbb{I}^n) \hookrightarrow L^{n'}(\mathbb{I}^n).$

Sobolev embeddings in r.i. spaces

Kerman and Pick studied the Sobolev embeddings among r.i. spaces. In particular, they solved the following problems:

- * Given an r.i. range space $X(\mathbb{I}^n)$, find the largest r.i. domain space, with a.c. norm, namely $Z(\mathbb{I}^n)$, satisfying

$$W_0^1 Z(\mathbb{I}^n) \hookrightarrow X(\mathbb{I}^n).$$

This means that if $W_0^1 \tilde{Z}(\mathbb{I}^n) \hookrightarrow X(\mathbb{I}^n) \Rightarrow \tilde{Z}(\mathbb{I}^n) \hookrightarrow Z(\mathbb{I}^n)$.

- * Given an r.i. domain space $Z(\mathbb{I}^n)$, describe the smallest r.i. range space, namely $X(\mathbb{I}^n)$, that verifies

$$W_0^1 Z(\mathbb{I}^n) \hookrightarrow X(\mathbb{I}^n).$$

That is, if $W_0^1 Z(\mathbb{I}^n) \hookrightarrow \tilde{X}(\mathbb{I}^n) \Rightarrow X(\mathbb{I}^n) \hookrightarrow \tilde{X}(\mathbb{I}^n)$.

Examples

Classical Sobolev embedding theorem
 $W_0^1 L^p(\mathbb{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathbb{I}^n), \quad 1 \leq p < n.$



Hunt; O'Neil; Peetre.
 $W_0^1 L^p(\mathbb{I}^n) \hookrightarrow L^{pn/(n-p), p}(\mathbb{I}^n).$



Kerman and Pick
 Optimal r.i. domain space: $L^p(\mathbb{I}^n).$

Kerman and Pick
 Optimal r.i. range space: $L^{pn/(n-p), p}(\mathbb{I}^n).$

Examples

Critical Sobolev embedding theorem

$$W_0^1 L^n(\mathbb{I}^n) \hookrightarrow L^p(\mathbb{I}^n), \quad 1 \leq p < \infty.$$



Maz'ya; Hansson; Brézis and Wainger

$$W_0^1 L^n(\mathbb{I}^n) \hookrightarrow L^{\infty, n; -1}(\mathbb{I}^n).$$



Kerman and Pick

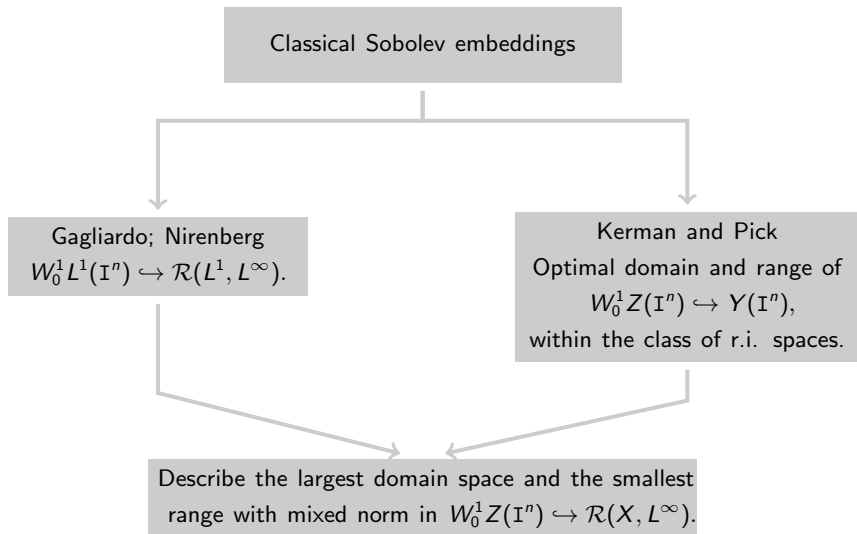
Optimal r.i. domain space: $Z_{L^{\infty, n; -1}}(\mathbb{I}^n)$.



Hansson; Kerman and Pick

Optimal r.i. range space: $L^{\infty, n; -1}(\mathbb{I}^n)$.

Motivation



Problem

Let $X(I^{n-1})$ be an r.i. space and let $Z(I^n)$ be an r.i. space, with a.c. norm. Our aim is to study the Sobolev embedding

$$W_0^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty). \quad (1)$$

We are interested in the following questions:

- Given a mixed norm space $\mathcal{R}(X, L^\infty)$, we want to find the largest r.i. domain space, with a.c. norm, satisfying (1).
- Let $Z(I^n)$ be an r.i. domain space, with a.c. norm. We would like to find the smallest range space of the form $\mathcal{R}(X, L^\infty)$ for which (1) holds.

Examples

Classical Sobolev embedding theorem
 $W_0^1 L^p(\mathbb{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathbb{I}^n), \quad 1 \leq p < n.$

$W_0^1 L^p(\mathbb{I}^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-p), p}, L^\infty).$

$L^p(\mathbb{I}^n)$ optimal r.i. domain space.

$\mathcal{R}(L^{p(n-1)/(n-p), p}, L^\infty)$ optimal
 range of the form $\mathcal{R}(X, L^\infty)$.

Examples

Critical Sobolev embedding theorem

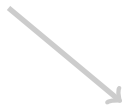
$$W_0^1 L^n(\mathbb{I}^n) \hookrightarrow L^p(\mathbb{I}^n), \quad 1 \leq p < \infty.$$



$$W_0^1 L^n(\mathbb{I}^n) \hookrightarrow \mathcal{R}(L^{\infty, n; -1}, L^\infty).$$



$Z_{L^{\infty, n; -1}}(\mathbb{I}^n)$ optimal r.i. domain space.



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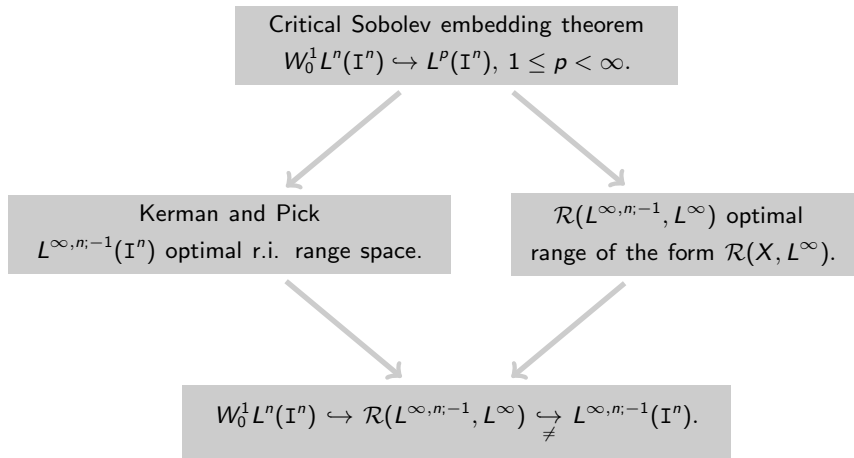
Domain space: $L^p(\mathbb{I}^n)$, $1 \leq p < n$

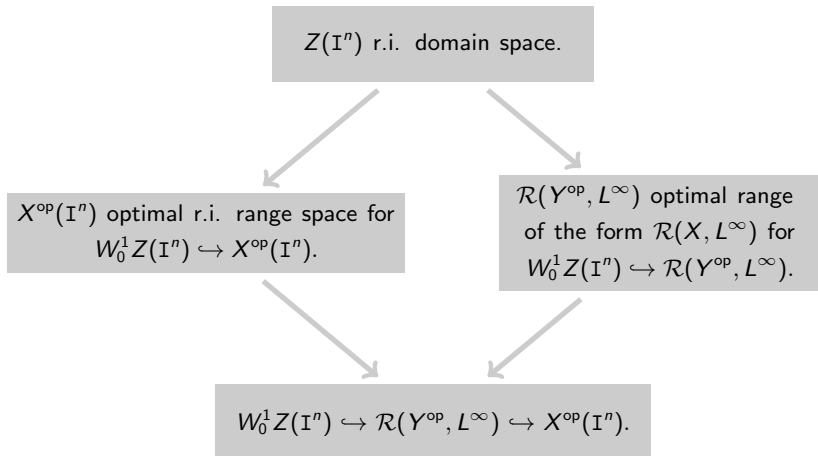
Classical Sobolev embedding theorem
 $W_0^1 L^p(\mathbb{I}^n) \hookrightarrow L^{pn/(n-p)}(\mathbb{I}^n)$, $1 \leq p < n$.

Kerman and Pick
 $L^{pn/(n-p), p}(\mathbb{I}^n)$ optimal r.i. range space.

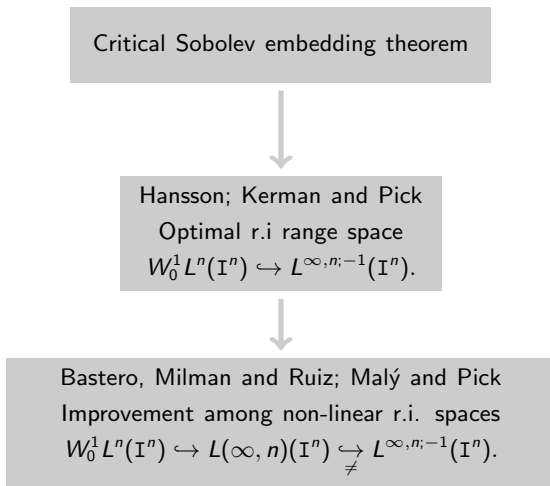
$\mathcal{R}(L^{p(n-1)/(n-p), p}, L^\infty)$ optimal
 range of the form $\mathcal{R}(X, L^\infty)$.

$W_0^1 L^p(\mathbb{I}^n) \hookrightarrow \mathcal{R}(L^{p(n-1)/(n-p), p}, L^\infty) \not\hookrightarrow L^{pn/(n-p)}(\mathbb{I}^n)$.

Domain space: $L^n(\mathbb{I}^n)$ 

Domain space: $Z(\mathbb{I}^n)$ 

Critical case of the classical Sobolev embedding



Critical case of the classical Sobolev embedding

Critical Sobolev embedding theorem



Optimal range of the form $\mathcal{R}(X, L^\infty)$
 $W_0^1 L^n(\mathbb{I}^n) \hookrightarrow L^{\infty, n; -1}(\mathbb{I}^n).$



Improvement among non-linear spaces of the form $\mathcal{R}(X, L^\infty)$
 $W_0^1 L^n(\mathbb{I}^n) \hookrightarrow \mathcal{R}(L(\infty, n), L^\infty) \not\hookrightarrow \mathcal{R}(L^{\infty, n; -1}, L^\infty).$

The end

Thank You!!