Fredholm theory for the Dirichlet problem

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“Harmonic Analysis and Partial Differential Equations”

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Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}.$$

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$$\lim_{\mathbb{D} \ni z \longrightarrow e^{i\theta}} u_f(z) = f(\theta)$$

for all $0 \leq \theta < 2\pi$.

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$\Omega = \mathbb{D} = \{|z| < 1\}$ $\quad$ $f = f(\theta) \in L^p(S^1)$, $1 \leq p \leq \infty$.

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for all nontangential cone $\Gamma(\theta)$, for all $0 \le \theta < 2\pi$.

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- $f : \partial\Omega \longrightarrow \mathbb{R}$ continuous.

Double layer potential

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- $\mathrm{d}\sigma$ is the surface measure of $\partial\Omega$.

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- Behaviour of $\mathcal{D}f$ on $\partial\Omega$?

Observations

Standard computations give

$$\frac{\partial}{\partial n_Q} R(X,Q) = c_n \frac{< X - Q, n_Q >}{|X-Q|^n}, \quad X \notin \partial\Omega,\ Q \in \partial\Omega.$$

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$$K(P,Q) = c_n \frac{< P - Q, n_Q >}{|P - Q|^n}$$

where $P, Q \in \partial\Omega$ and $P \neq Q$.

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- $Tf \in \mathcal{C}(\partial\Omega)$.

Behaviour of $\mathcal{D}f$ in the boundary

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Take $f \equiv 1$. Then

$$\mathcal{D}f(X) = \int_{\partial\Omega} \frac{\partial}{\partial n_Q} R(X,Q) \mathrm{d}\sigma(Q) = \begin{cases} 1 \text{ if } X \in \Omega, \\ 0 \text{ if } X \notin \overline{\Omega}, \end{cases}$$

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When $f \equiv 1$,

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Theorem

For all $f \in \mathcal{C}(\partial\Omega)$,

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- Denote $\mathcal{D}_+$ the restriction to $\partial\Omega$ of the extension from inside.
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The operator $T : \mathcal{C}(\partial\Omega) \longrightarrow \mathcal{C}(\partial\Omega)$ is compact.

Fredholm Alternative

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Let H_1 be a Hilbert space and $T : H_1 \longrightarrow H_1$ a linear, bounded and compact operator. For $\lambda \neq 0$, TFAE:

- $T - \lambda I$ is surjective.
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$$\mathcal{D}_+ = \frac{1}{2}I + T \quad \left(\lambda = -\frac{1}{2}\right)$$

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The adjoint of T is

$$T^* f(P) = \int_{\partial\Omega} K^*(P,Q) f(Q) \mathrm{d}\sigma(Q),$$

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- Goal: see that $T^* + \frac{1}{2}I$ is injective.

Single Layer Potential

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is a diffeomorphism. For $P \in \partial\Omega$ and $|t| < \epsilon$, $t \neq 0$, we define

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If $f\in\mathcal{C}$,

- $DSf\in\mathcal{C}(\overline{\Omega\cap V})$.
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$$\varphi(x) = \varphi(y) + < x - y, \nabla\varphi(y) > + e(x, y), \tag{1}$$

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Thank you for your attention!