Invariant Manifolds Near L_1 and L_2 Points in the Restricted Three-Body Problem

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BGSMath BARCELONA GRADUATE SCHOOL OF MATHEMATICS

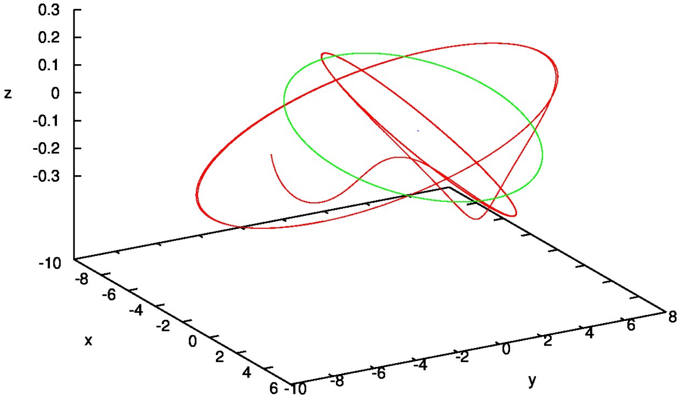
EXCELENCIA MARÍA DE MAEZTU

TraX Stability and Transitions in Physical Processes

Summary

The non-trivial Oterma's Dynamics

The comet 39P/Oterma have an interesting and intriguing dynamics.
Its orbit is mainly perturbed by Jupiter so that, sometimes its trajectory is in between Jupiter and Saturn and sometimes between Jupiter and Mars.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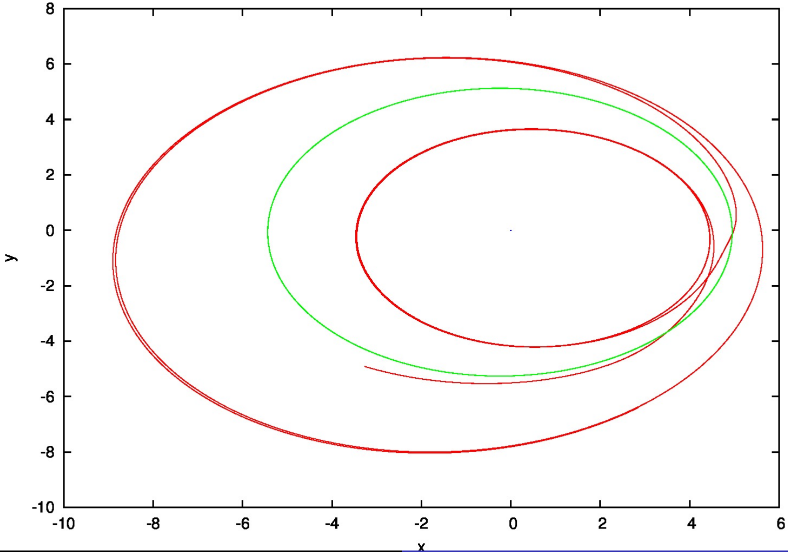
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x

- The n-body problem;
- The (planar circular) restricted three-body problem;
- Sidereal $\times$ synodical systems of coordinates;
- Equilibrium points;
- Dynamics in energy levels;
- Zero-velocity curves;
- Lyapunov orbits;

V. SZEBEHELY. *Theory of Orbits. The Restricted Problem of Three Bodies.* Academic Press. Nem York. 1967.

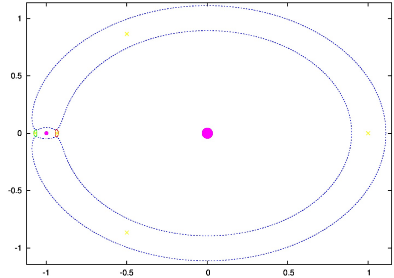
In this work, all the computations and analysis will be done under the assumption that the problem is planar and circular, i.e., that Oterma moves in the same plane as Jupiter and Sun and that they describe a circular moviment.

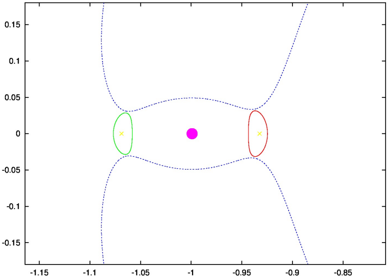
With this, the Hamiltonian of this problem is given by:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

where $r_1 = (x-\mu)^2 + y^2$ and $r_2 = (x+1-\mu)^2 + y^2$.

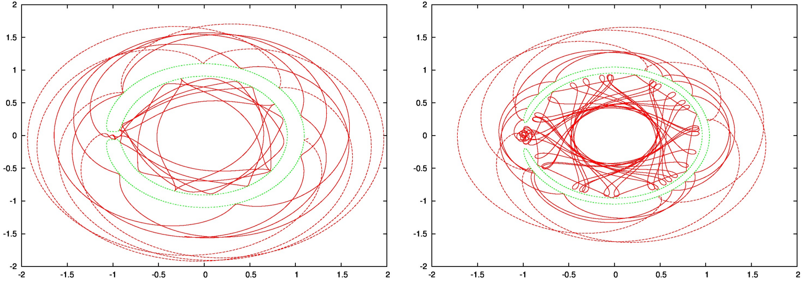
A possible scenario for this jump is:





W. S. KOON, M. W. LO, J. E. MARSDEN, S. D. ROSS. *Resonance and Capture of Jupiter Comets.* Celestial Mechanics and Dynamical Astronomy **81**: 27-38, 2001.

As examples:



We would like to obtain an integrable approximation to the Hamiltonian near the L_1 and L_2 points.

Here, we will use the fact that those points are of the type centre $\times$ saddle and compute normal forms.

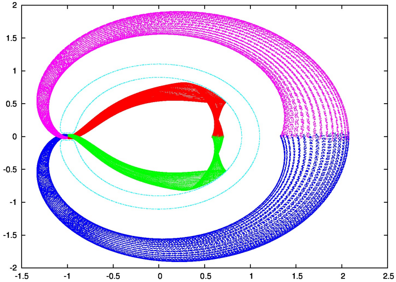
By doing so, we will get an integrable approximation to the Hamiltonian function.

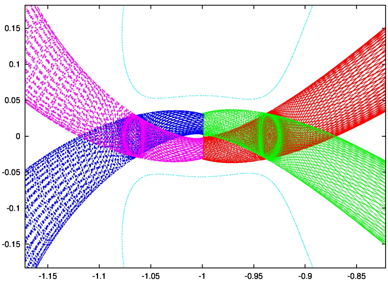
Using a computer algebra system (end of this slide) it is possible to compute such an approximation in high-degree polynomials, being their degrees an input to the algorithm.

With this approximation in hands, it is not difficult to compute an approximation to the dynamical objects: periodic orbits, stable/unstable manifolds, and so on.

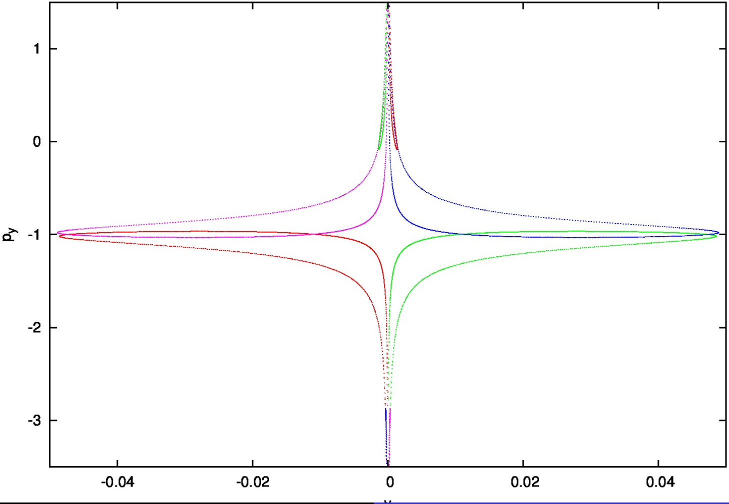
À. JORBA. *Numerical Computation of Normal Forms, Centre Manifolds and First Integrals of Hamiltonian Systems.* Experimental Mathematics, Vol. 8 (1999). No. 2.

Zero-velocity curves, equilibrium points, periodic orbits and stable/unstable manifolds of those orbits computed by the computer algebra system:

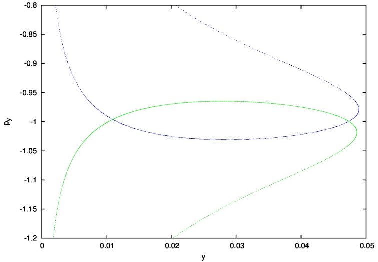




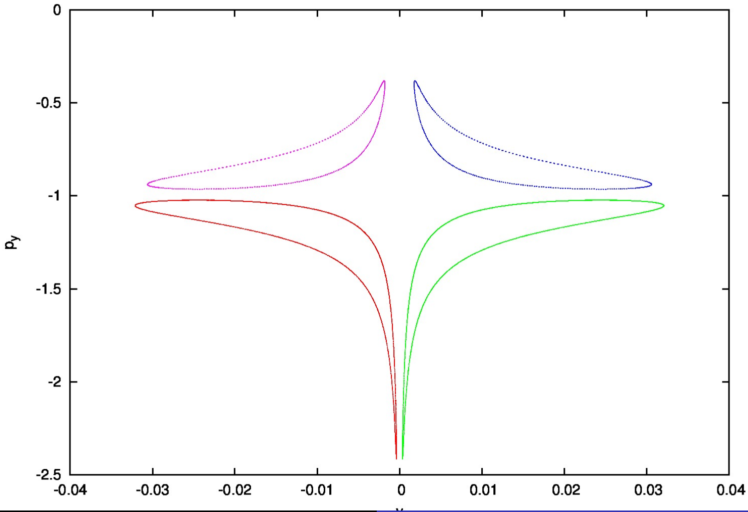
There are energy levels where those manifolds intersect:



(Zooms in some regions:)

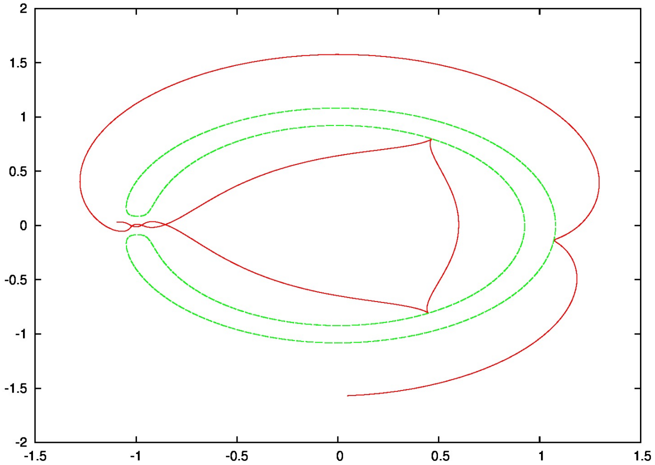


And there are levels where they do not cross:



Originally, the Sun-Jupiter-Oterma system is a three-dimensional system and elliptic, so, in order to adjust the real data to the planar circular model, we do the following:

- Project Oterma in the plane where Sun and Jupiter move;
- Rotate such plane so that it is now the xy plane;
- Inside that plane, rotate x and y axes in such a way that both Jupiter and Sun are in the x axis;
- Apply a change in the units of measure of position and velocity so that Jupiter is fixed in $(-1+\mu, 0, 0)$, Sun in $(\mu, 0, 0)$ and Jupiter's period of revolution is 2π.



Remark: There is a small adjustment in Oterma's velocity that still needs a more careful investigation!

One may consider that the jump experimented by Oterma can be explained by the planar circular model, however, when dealing with real data, gathered from JPL Horizons system (https://ssd.jpl.nasa.gov/horizons.cgi) and adjusting them to this model, the results are not satisfying, because in order to have the same behaviour (jump) it is necessary to do a (still misterious) adjustment.

In other words, **qualitatively** the planar circular model is suitable, yet **quantitatively** maybe it is not the best one.

The tools presented here, in addition to the comprehension of natural phenomena (Oterma is not the only comet that exibit this jump behaviour), can be applied to the design of space missions, as the particle may be considered to be, for instance, a probe designed to navigate through some planet/satellite. In this case, it can take advantage of these manifolds to follow its way spending the less fuel possible getting itself into an interesting region.

- **Elliptical model**;
- Spatial model;
- Combining elliptical and spatial models;
- Bicircular model;
- etc.

The Elliptic model

Now, we shall move to a, let us say, new problem, but based on the already studied one: The Planar **Elliptical** Restricted Three-Body Problem.
The system of coordinate now is of the same type as the circular problem, but, instead of a rotating frame, we will use a roto-pulsating one, so that the primaries are located at the same place.

The Elliptic model

This model is based in the following Hamiltonian:

$$\begin{aligned} H(x, y, p_x, p_y, f) &= \frac{1}{2}\left((p_x + y)^2 + (p_y - x)^2\right) \\ &\quad - \frac{1}{1 + e\cos f}\left(\frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right), \end{aligned}$$

where r_1 and r_2 are defined as above ($r_1 = (x-\mu)^2 + y^2$ and $r_2 = (x + 1 - \mu)^2 + y^2$), e is Jupiter's eccentricity and f is its true anomaly.

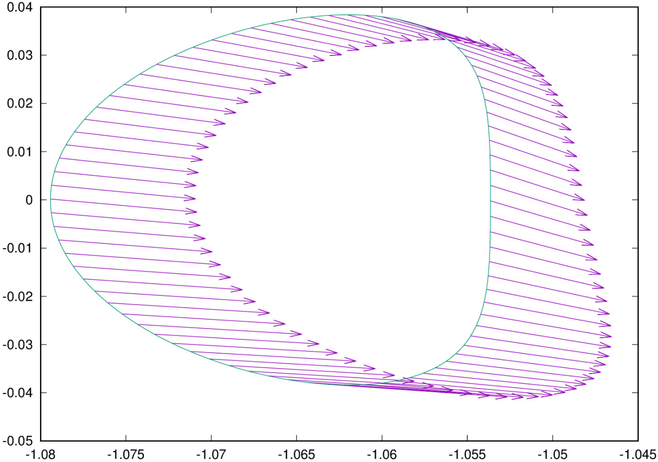
This model can be seen as a 2π-periodic perturbation of the circular one, depending on the eccentricity of the moviment. In the case of Jupiter's one ($e \approx 0.0489$) it is reasonable to see it in this way.

The equilibria points, in these roto-pulsating coordinates, are the same as in the circular model.
The periodic orbits around L_1 and L_2 in the circular model now turn to be invariant tori, with the same stability (centre $\times$ saddle).

Given a periodic orbit previously calculated to the circular model, it is possible to use it as a seed to start the computations of the invariant tori in the elliptic model.

We start by defining an autonomous map f from the phase space to itself as the time-2π flow.

So, we can now look for a parametrization x of the torus such that $f(x(\theta)) = x(\theta + \omega)$, where ω is its rotation number.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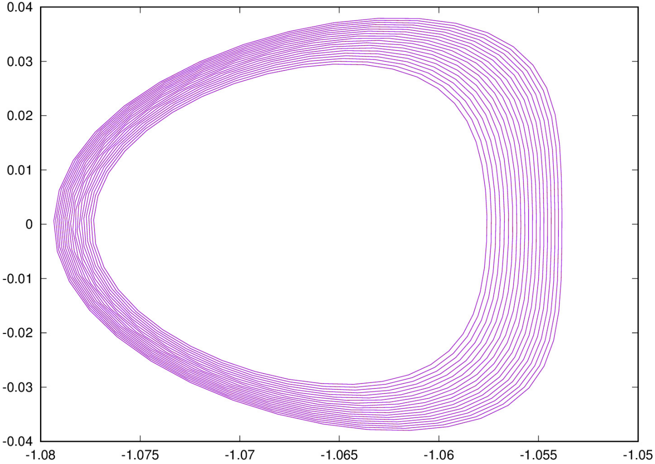
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$$\omega = 12.293631302025044 9$$

$$\omega = 12.193631302025044 9$$

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