Postquantum Cryptography: what, why, and how?

SIMBA

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How can they agree on a secret key to encrypt their communications, using e.g. AES?

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Bob chooses a private key $1 < b < p$, and publishes $B = \alpha^b \mod p$.
They may use the **shared secret** $A^b \equiv B^a \equiv \alpha^{ab} \mod p$.

Computational problems

Problem (Discrete Logarithm - DLP)

Given a cyclic group $G = \langle \alpha \rangle$ and an element $\beta \in G$, find $x \in \mathbb{Z}$ such that $\beta = \alpha^x$.

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Problem (Diffie-Hellman - DHP)

Given a cyclic group $G = \langle \alpha \rangle$ *and elements* $\alpha^a, \alpha^b \in G$*, find* α^{ab}*.*

Introduction: Diffie-Hellman

Why? Solving the DLP

What? Postquantum Cryptography

How? Isogenies and SIDH

Why? Solving the DLP

Let's see some algorithms to solve for discrete logarithms!

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2. Compute $\beta\alpha^{-am}$, for $0 \leq a < m$, and check for a match $\beta\alpha^{-am} = \alpha^b$.
3. If so, $\beta = \alpha^{am+b}$ and $x = am + b$.

Pohlig-Hellman

Idea: factor $N = \prod_{i=1}^{r} p_i^{e_i}$, and obtain $x \mod p_i^{e_i}$ for each i. Then use the Chinese Remainder Theorem to combine the information.

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If $p^e \mid N$, then α^{N/p^e} has order p^e, and $\beta^{N/p^e} = \left(\alpha^{N/p^e}\right)^x$. We can compute $x \mod p^e$!

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*Only useful if N is smooth (all prime factors are small).

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2. Find a relation of the form $\alpha^k \beta = \prod_{i=1}^{t} g_i^{e_i}$.
3. The discrete logarithm will be

$$x = \log_\alpha(\beta) = \sum_{i=1}^{t} e_i \log_\alpha(g_i) - k = \sum_{i=1}^{t} e_i y_i - k.$$

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If $t = 0$, then $L_n[0, \gamma] = (\log n)^{\gamma + o(1)}$ is polynomial in $\log n$.
If $t = 1$, then $L_n[1, \gamma] = n^{\gamma + o(1)}$ is exponential in $\log n$.

Summary of complexities

Algorithm	**Complexity**
Exhaustive search	$O(N)$
Baby step – giant step	Time $O(\sqrt{N})$, memory $O(\sqrt{N})$
Pohlig-Hellman	$O(\sum_{i=1}^{r} e_i(\log N + \sqrt{p_i}))$
Index calculus in $\mathbb{F}_{p^n}$	$L_{p^n}[1/2, \sqrt{2}]$
NFS-DLP in $\mathbb{F}_{p^n}$	$\mathbf{L_{p^n}[1/3, c]}$

Table: Algorithms solving DLP in a group of order $N = \prod_{i=1}^{r} p_i^{e_i}$.

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Why Post-Quantum Cryptography, then?

PQCRYPTO EU-Project

"The EU and governments around the world are investing heavily in building quantum computers; society needs to be prepared for the consequences, including cryptanalytic attacks accelerated by these computers." [Lan15]

Why Post-Quantum Cryptography, then?

NIST's Report on Post-Quantum Cryptography
"Some experts even predict that within the next 20 or so years, sufficiently large quantum computers will be built to break essentially all public key schemes currently in use." [Moo+16]

Introduction: Diffie-Hellman

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1. It must be efficient to use with existing hardware.
2. It must be resistent both to classical and quantum adversaries.

What do we need to develop?

We can't use ciphers based on **discrete logarithms** (Diffie-Hellman) or **integer factorization** (RSA). That is, we need to look for new kinds of asymmetric encryption.

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However, "symmetric algorithms [...] should be usable in a quantum era", because breaking them usually involves brute-force search in the key space, and "doubling the key size will be sufficient to preserve security" [Moo+16].

What techniques are involved in PQ Cryptography?

- Lattice-based cryptography
- Code-based cryptography
- **Isogeny-based cryptography**

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Elliptic curves

Let K be a field of characteristic different from $2, 3$, and $A, B \in K \subseteq L$ with $4A^3 + 27B^2 \neq 0$. An **elliptic curve** E is the set of points (x, y) that satisfy the equation

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$$E\colon y^2 = x^3 + Ax + B.$$

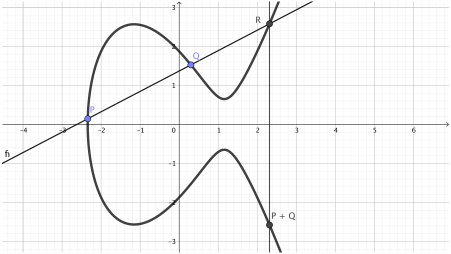
More precisely, we define the set of L-rational points,

$$E(L) := \{(x, y) \in L \times L \mid y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}.$$

In homogeneous coordinates, the equation is $y^2 z = x^3 + Axz^2 + Bz^3$, and $\mathcal{O} = (0 : 1 : 0)$ is the only **point at infinity**.

Elliptic curves are groups

Given two points $P, Q \in E(K)$, we define an operation on the points:



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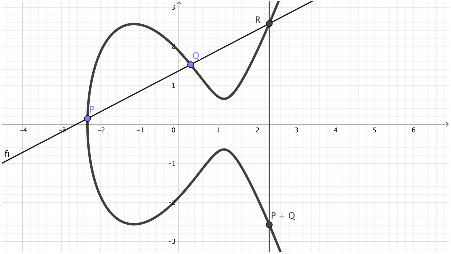

Theorem

The set $E(K)$ with the operation $+$ is an abelian group.

The j-invariant

Given a curve E: $y^2 = x^3 + Ax + B$, its j-invariant is

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Two curves are isomorphic over $\bar{K}$ if and only if they have the same j-invariant.
For each $j_0 \in \bar{K}$, there exists a curve E with $j(E) = j_0$.

Isogenies

Given two elliptic curves E_1, E_2 over K, an **isogeny** between them is a non-constant map

$$\phi \colon E_1(\bar{K}) \to E_2(\bar{K})$$

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Isogenies can be put in a **standard form**:

$$\phi(x, y) = \left(\frac{p(x)}{q(x)}, y\frac{s(x)}{t(x)} \right)$$

Multiplication by n

The *multiplication-by-n* map $[n]\colon E \to E$ is an isogeny for all non-zero $n \in \mathbb{Z}$. Its kernel is written as $E[n]$, the **group of n-torsion points**.

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Let $p = \operatorname{char} K$. For any prime $\ell \neq p$, we have $E[\ell^n] \cong \mathbb{Z}/\ell^n\mathbb{Z} \times \mathbb{Z}/\ell^n\mathbb{Z}$. This group has $\ell^{n-1}(\ell+1)$ cyclic subgroups of order ℓ^n.

Quotient curve

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Theorem

Let E_1 be an elliptic curve over K, and let G be a finite subgroup of $E_1(\bar{K})$. There exist a curve E_2 and an isogeny $\phi\colon E_1 \to E_2$, such that $\ker \phi = G$. Moreover, ϕ and E_2 are unique up to isomorphism.

We will write $E_2 = E_1/G$.

Hasse's theorem

Theorem

Let E be an elliptic curve defined over a finite field $\mathbb{F}_q$, $q = p^r$. The number of $\mathbb{F}_q$-rational points of E is

$$\#E(\mathbb{F}_q) = q + 1 - t,$$

with $|t| \leq 2\sqrt{q}$.

Supersingular curves

Theorem

Let E be a curve over a finite field $\mathbb{F}_q$, $q = p^r$. TFAE:

- *E is supersingular.*
- $E[p] = \{\mathcal{O}\}$.
- *$[p]$ is purely inseparable.*
- $\#E(\mathbb{F}_q) = q + 1 - t$, *with* $t \equiv 0 \mod p$.
- *$End(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a quaternion algebra.*

Given a prime p, there are about $p/12$ supersingular elliptic curve isomorphism classes defined over $\bar{\mathbb{F}}_p$.

Supersingular Isogeny Diffie Hellman - Setting

Let $p = 2^{e_A}3^{e_B} - 1$ be a prime with $2^{e_A} \approx 3^{e_B}$, set $K = \mathbb{F}_{p^2}$.

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The curve $E_0 \colon y^2 = x^3 + x$ is supersingular, and

$$\#E_0(\mathbb{F}_{p^2}) = (p+1)^2 = (2^{e_A}3^{e_B})^2.$$

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We have $E_0[2^{e_A}] = \langle P_A, Q_A \rangle,\ E_0[3^{e_B}] = \langle P_B, Q_B \rangle \subset E_0(\mathbb{F}_{p^2})$.

SIDH - Private keys

Alice chooses a pair $(m_A, n_A) \in \mathbb{Z}/2^{e_A}\mathbb{Z} \times \mathbb{Z}/2^{e_A}\mathbb{Z}$ (not both divisible by 2). This is her private key.

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Bob chooses a pair $(m_B, n_B) \in \mathbb{Z}/3^{e_B}\mathbb{Z} \times \mathbb{Z}/3^{e_B}\mathbb{Z}$ (not both divisible by 3). This is his private key.

SIDH - Key exchange

$$
\begin{tikzcd}
 & E_0/\langle [m_A]P_A + [n_A]Q_A \rangle \\
E_0 \arrow[ur, "\phi_A"] \arrow[dr, "\phi_B"'] & \\
 & E_0/\langle [m_B]P_B + [n_B]Q_B \rangle
\end{tikzcd}
$$

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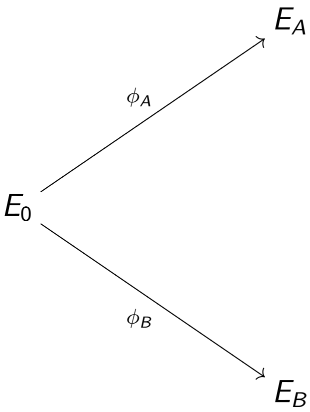

SIDH - Key exchange

$$
\begin{tikzcd}
& E_A \arrow[dd, dashed, shift left, "{\begin{matrix} E_A \\ \phi_A(P_B) \\ \phi_A(Q_B) \end{matrix}}"] \\
E_0 \arrow[ur, "\phi_A"] \arrow[dr, "\phi_B"'] & \\
& E_B \arrow[uu, dashed, shift left, "{\begin{matrix} E_B \\ \phi_B(P_A) \\ \phi_B(Q_A) \end{matrix}}"]
\end{tikzcd}
$$

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\begin{tikzcd}
 & E_A \arrow[dd, dashed, shift left, "{E_A,\ \phi_A(P_B),\ \phi_A(Q_B)}"] \arrow[dr, "\phi_B'"] & & \\
E_0 \arrow[ur, "\phi_A"] \arrow[dr, "\phi_B"'] & & E_{AB} \arrow[r, squiggly] & j_{AB} \\
 & E_B \arrow[uu, dashed, shift left, "{E_B,\ \phi_B(P_A),\ \phi_B(Q_A)}"] \arrow[ur, "\phi_A'"'] & &
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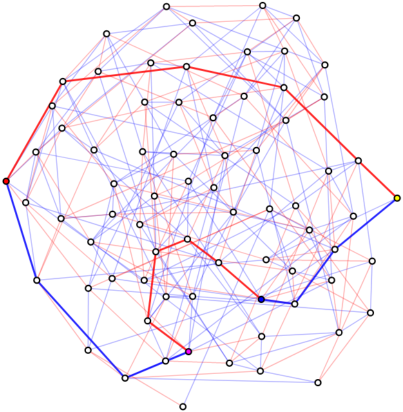

Figure: SIDH graph with $p = 2^5 3^3 - 1 = 863$.

Computational problems

Problem (Supersingular Isogeny problem (CSSI))

Let $\phi_A \colon E_0 \to E_A$ be an isogeny with kernel $\langle [m_A]P_A + [n_A]Q_A \rangle$, where m_A, n_A are chosen randomly in $\mathbb{Z}/\ell_A^{e_A}\mathbb{Z}$ and not both divisible by ℓ_A.
Given the curves E_0, E_A and the values $\phi_A(P_B)$ and $\phi_A(Q_B)$, find a generator R_A of $\langle [m_A]P_A + [n_A]Q_A \rangle$.

Analog to DLP in the Diffie-Hellman setting.

Computational problems

Problem (Supersingular D.-H. problem (SSCDH))

Let

$$\begin{cases} \phi_A\colon E_0 \to E_A = E_0/\langle [m_A]P_A + [n_A]Q_A \rangle, \\ \phi_B\colon E_0 \to E_B = E_0/\langle [m_B]P_B + [n_B]Q_B \rangle \end{cases}$$

be isogenies defined as in the SIDH protocol.
Given the curves E_A, E_B and the points $\phi_A(P_B)$, $\phi_A(Q_B)$, $\phi_B(P_A)$, $\phi_B(Q_A)$, find the j-invariant of the curve

$$E_0/\langle [m_A]P_A + [n_A]Q_A, [m_B]P_B + [n_B]Q_B \rangle.$$

Analog to DHP in the Diffie-Hellman setting.

SIDH security

- The same problems in the **ordinary** case (e.g., non-supersingular) can be solved with a quantum computer in subexponential time.
- The best strategy to break SIDH is almost brute-force, at $O(\sqrt[4]{p})$ and $O(\sqrt[6]{p})$ (exponential in $\log p \sim e_A, e_B$).
- It looks like the **auxiliary points** ($\phi_A(P_B)$ and so on) are revealing too much information, but so far nobody* has been able to exploit them.

SIDH/SIKE in production

KEM	Public Key size (bytes)	Ciphertext (bytes)	Secret size (bytes)	KeyGen (op/sec)	Encaps (op/sec)	Decaps (op/sec)	NIST level
HRSS-SXY	1138	1138	32	3952.3	76034.7	21905.8	1
SIKE/p434	330	346	16	367.1	228.0	209.3	1

Figure: Comparison between lattice-based HRSS-SXY and isogeny-based SIKE [Kwi19].

SIDH/SIKE in production

Figure: Ostrich vs turkey [KV19].

Conclusions

- Public-key cryptosystems based in RSA and Diffie-Hellman could be broken in a few years.
- Current efforts in finding and **testing** new postquantum standards.
- SIDH/SIKE is the most prominent isogeny-based cryptography proposal, *however* there are other constructions to explore (CGL, CSIDH, higher genus...).

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Thank you!