Elliptic PDEs 2: Existence and Uniqueness of Solutions

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20th May 2020

How do we solve PDEs?

- Find a space of functions where existence and uniqueness are viable.
- Prove that, in fact, the solutions are in more desirable spaces.

For example, a good space to have existence and uniqueness is L^2 or H^1, and a good space to have finally the solutions could be C^1 or $\mathcal{C}^\infty$.

Introduction: Weak Topology

Definition

Let E be a Banach space, and E' the space of continuous linear functionals from E to $\mathbb{R}$. We say f_n is weak convergent to f in E if, for any $\phi \in E'$, $\phi(f_n) \to \phi(f)$. We note $f_n \rightharpoonup f$.

Example

$E = L^2(-\pi, \pi)$, $f_n(x) = \sin(nx)$. Let $\phi \in E'$. $\phi(f_n) = \int_{-\pi}^{\pi} f_n g$, $g \in L^2(-\pi, \pi)$, therefore $g \in L^1(-\pi, \pi)$. By the Riemann-Lebesgue lemma, $\phi(f_n) \to 0$. Hence we say $f_n \rightharpoonup 0$.

Introduction: Compact Operators

Definition

Let E, F be Banach spaces, a (linear) operator $T : E \to F$ is said to be compact, if $\overline{T(B_E)}$ is a compact set in F.

Example

- Finite rank operators are compact.
- *Integral* operators are usually compact.

Proposition

Let $T : E \to F$ be compact, and let $f_n \rightharpoonup f$ in E. Then, $T(f_n) \to T(f)$ in the strong (norm) topology of F.

Introduction: $H^1(\Omega)$

Definition

Let Ω be a Lipschitz domain.

$$H^1 = \{f : \Omega \to \mathbb{R} \text{ such that } |f|, \|\nabla f\| \in L^2(\Omega)\}$$

- $H^1(\Omega)$ is a Hilbert space, with the product

$$< f, g >= \int_\Omega fg + \nabla f \cdot \nabla g$$

- $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega)$ is well defined, continuous and compact, and for $u \in \mathcal{C}^1(\Omega)$, $\mathrm{Tr}\, u = u|_{\partial\Omega}$.
- $i : H^1(\Omega) \to L^2(\Omega)$, the inclusion, is compact.

Definition

$$H_0^1 = \{f \in H^1(\Omega) \text{ such that } \mathrm{Tr}(f) = 0\}$$

Introduction: Maximum Principle

Theorem (Maximum principle in weak form)

Let $u \in H^1(\Omega)$ $\begin{cases} -\Delta u \geq 0 \;\; in \; \Omega, \\ u \geq 0 \;\; on \; \partial\Omega. \end{cases}$ *Then,* $u \geq 0 \;\; in \; \Omega.$

We have to understand $-\Delta u \geq 0$ in the weak sense, this is, integrating by parts and taking $v \in H_0^1(\Omega), v \geq 0$,

$$\int_\Omega -\Delta u \cdot v = \int_\Omega \nabla u \cdot \nabla v \geq 0$$

Proof

Set $u = u^+ - u^-$, where $u^+, u^- \geq 0$ and $u^+ u^- = 0$.
$\nabla u = \nabla u^+ - \nabla u^-$. Then, putting $v = u^-$ (notice $\mathrm{Tr}(u^-) = 0$, and then $u^- \in H_0^1(\Omega)$),

$$\int_\Omega \nabla u \cdot \nabla u^- = -\int_\Omega \|\nabla u^-\|^2 \geq 0$$

Then $\nabla u^- = 0$, so $u^- = 0$ and finally we have $u = u^+ \geq 0$.

The Laplacian: The Dirichlet Problem

Find a function u in Ω that is a solution of $\Delta u = 0$ with a prescribed boundary value $u = g$ in $\partial\Omega$.

Depending on the methods used, the space of functions and the exact definitions of $\partial\Omega$ can be different.

The Laplacian: Energy Methods

Let Ω be a bounded Lipschitz domain. Let $g \in L^2(\partial\Omega)$. Let $H^1_g(\Omega) = \{f \in H^1(\Omega) \text{ such that } \operatorname{Tr}(f) = g\}$.

We define the *Dirichlet energy* of a function:

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$$

If we find a minimizer v of E in $H^1_g(\Omega)$, then v will be a solution for the Dirichlet problem.

The Laplacian: Energy Methods 2

A necessary condition for a minimizer v is, for all $\varphi \in H_0^1(\Omega)$,

$$E(v) \leq E(v + t\varphi) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} E(v + t\varphi) = 0 \text{ at } t = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v + t\nabla\varphi|^2 = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (2t\nabla v \cdot \nabla\varphi + t^2|\nabla\varphi|^2) =$$

$$= 2\int_\Omega \nabla v \cdot \nabla\varphi = -2\int_\Omega \Delta v \varphi$$

$$\Delta v = 0$$

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Theorem (Existence and uniqueness)

Let Ω be a bounded Lipschitz domain, and suppose $H^1_g \neq \emptyset$. Then, there exists a unique solution v for the Dirichlet problem.

Existence

Let E_0 be the infimum of $E(w)$ in $w \in H^1_g(\Omega)$. Take $u_k \in H^1_g(\Omega)$ such that $E(u_k)E_0$. As ∇u_k is uniformly bounded in L^2, u_k are uniformly bounded in H^1, and then there is a partial subsequence convergent in L^2 and weakly convergent in H^1 to a limit v, because the inclusion is compact. Also, Tr is compact, so $\text{Tr}(v) = g$ as well. By Fatou's lemma, $E(v) \leq \liminf E(u_k)$, and then v is a minimizer.

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Uniqueness (energy)

The functional E is strictly convex, hence it can only attain one minimum because H^1_g is a convex set. This implies the minimizer must be unique.
If v is a solution, then putting $u \in H^1_0(\Omega)$,

$$E(v+u) = \int_\Omega \|\nabla v\|^2 + 2\nabla v \cdot \nabla u + \|\nabla u\|^2 = E(v) + 0 + E(u)$$

Hence $E(v) \leq E(v+u)$ for all suitable u and v is a minimizer. Then the solution is unique.

Uniqueness (comparison)

Suppose there are two solutions v_1, v_2. Consider the function $w = v_1 - v_2 \in H^1_0(\Omega)$, that satisfies $\Delta w = 0$. We can apply the maximum principle to w and $-w$, and we get $w = 0$.

The Laplacian: Viscosity Methods

Definition

A function $u \in \mathcal{C}(\overline{\Omega})$ is called *subharmonic* if for every $v \in \mathcal{C}^2$ that touches it from above in $x_0 \in \Omega$, meaning $v \geq u$ in Ω, $v(x_0) = u(x_0)$, $\Delta v(x_0) \geq 0$.

Example

- $u(x) = |x|$ is subharmonic in $\mathbb{R}$.
- $u : [-1, 1] \to \mathbb{R}$, $u(x) = 1 - \chi_{[-1,0]}$ satisfies the condition but it is not subharmonic because it is not continuous.

We say a continuous function w is *superharmonic* if $-w$ is subharmonic.

We say a continuous function w is *harmonic* if it is subharmonic and superharmonic.

The Laplacian: Viscosity Methods 2

Proposition

If u_1, u_2 are subharmonic, $\max(u_1, u_2)$ is subharmonic as well.

Easy part of the proof of existence

Take the set $S = \{u \in \mathcal{C}(\overline{\Omega})$ such that u is subharmonic$\}$.

$$v(x) = \sup_{u \in S} u(x)$$

Then v is subharmonic because it is a supremum of subharmonic functions.

The hard part is proving that v is actually harmonic.

More General Elliptic Operators

Elliptic PDEs are generalizations of the Laplace and Poisson equations, involving operators related to the Laplacian.

$$\Delta u = \mathrm{Tr}(D^2 u) = \mathrm{Div}(\nabla u).$$

This identity yields two natural generalizations of the Laplace operator, named in nondivergence and divergence form.

$$\mathcal{L}u = \mathrm{Tr}(A(x)D^2 u), \quad \mathcal{L}u = \mathrm{Div}(A(x)\nabla u).$$

$$\exists \lambda, \Lambda > 0 \text{ such that } \lambda I \leq A(x) \leq \Lambda I$$

Divergence Form Operators

Energy methods are preferred, because we can integrate by parts. We can recall what we did with the Laplacian and the proofs are similar. Recall

$$\int_\Omega -\operatorname{Div}(A(x)\nabla u)v = \int_\Omega A(x)\nabla u \cdot \nabla v = \int_\Omega (\nabla v)^\top A(x)\nabla u$$

Nondivergence Form and Pucci Operators

Energy methods cannot be adapted. We need viscosity methods.

Definition

Let $0 < \lambda < \Lambda$. We define the Pucci operators with the ellipticity constants λ, Λ, where M is a $n \times n$ symmetric matrix:

$$\mathcal{M}^+(M) = \Lambda \sum_{\mu_i > 0} \mu_i - \lambda \sum_{\mu_i < 0} \mu_i,$$

$$\mathcal{M}^-(M) = \Lambda \sum_{\mu_i < 0} \mu_i - \lambda \sum_{\mu_i > 0} \mu_i$$

where μ_i are the eigenvalues of M.

Nondivergence Form and Pucci Operators 2

Proposition

For any u such that D^2u exists,

$$\mathcal{M}^-(D^2u) \leq \mathcal{L}u \leq \mathcal{M}^+(D^2u)$$

Then the equation $\mathcal{L}u = 0$ can be relaxed to $\mathcal{M}^+ D^2u \geq 0$, $\mathcal{M}^- D^2u \leq 0$. In certain proofs this is preferred to get rid of $A(x)$.

The Obstacle Problem

Minimize the *energy* of a surface $v : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ above an obstacle φ.

$$\min \int_\Omega |\nabla v|^2, \quad v \geq \varphi.$$

This is equivalent to a nonlinear elliptic PDE problem:

$$\begin{cases} v \geq \varphi \text{ in } \Omega, \\ \Delta v \leq 0 \text{ in } \Omega, \\ \Delta v = 0 \text{ in the set } v > \varphi. \end{cases}$$

The Obstacle Problem: Energy Method

The proof is basically the same as for the unrestricted minimal surface, i.e., the Laplace equation. We have to change $H^1_g(\Omega)$ by the set $\{v \in H^1_g \text{ such that } v \geq \varphi\}$.

For the proof of uniqueness, we can recover the energy method or adapt the proof by the comparison principle.

The Obstacle Problem: Viscosity Method

We must consider the least supersolution of the problem, that is, define the pointwise infimum of all functions $w \in H^1_g$ satisfying

$$\begin{cases} w \geq \varphi \text{ in } \Omega, \\ \Delta w \leq 0 \text{ in } \Omega. \end{cases}$$

Then we can show that this function is actually harmonic when $v > \varphi$.

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