The reduction method: Determining the associated order in Hopf Galois structures of p-adic fields extensions

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Theorem

L/K is Galois if and only if $(1,\rho_G)$ is an isomorphism of K-vector spaces.

Definition

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A Hopf Galois structure of a finite extension of fields L/K is a pair $(H, \cdot)$, where H is a K-Hopf algebra and $\cdot\colon H \otimes_K L \longrightarrow L$ is a K-linear action, such that:

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L/K finite separable extension

$$\begin{tikzcd} L \arrow[d, dash] \\ K \end{tikzcd}$$

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Definition

$N \leq \mathrm{Perm}(X)$ is regular if for every $x, y \in X$ there is an unique $\eta \in N$ such that $\eta(x) = y$.

Theorem (Greither-Pareigis)

There is an one-to-one correspondence between:

$$\{(H,\cdot)\,|\,(H,\cdot) \textit{ Hopf Galois structure of } L/K\},$$

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If N is regular and G-stable, its corresponding Hopf Galois structure is

$$H = \widetilde{L}[N]^G = \{x \in \widetilde{L}[N]\,|\,\sigma(x) = x \text{ for all } \sigma \in G\}.$$

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We say that $(H,\cdot)$ is of type $[N]$.

$$\begin{array}{c} L \\ | \\ K \\ | \\ \mathbb{Q}_p \end{array}$$

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Theorem (Normal basis theorem)

There is a primitive element $\alpha \in L$ such that $\{\sigma(\alpha) \mid \sigma \in G\}$ is a K-basis of L.

$$\begin{array}{ccc} L & \text{———} & \mathcal{O}_L \\ \Big| G & & \Big| \\ K & \text{———} & \mathcal{O}_K \\ \Big| & & \Big| \\ \mathbb{Q}_p & \text{———} & \mathbb{Z}_p \end{array}$$

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Three kind of problems:

- Compute a basis of the associated order $\mathfrak{A}_H$.
- Is $\mathcal{O}_L$ free as $\mathfrak{A}_H$-module?
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Unique Hopf Galois structure of $L/\mathbb{Q}_3$: H with $\mathbb{Q}_3$-basis

$$w_1 = \mathrm{Id} \qquad w_2 = (\sigma - \sigma^{-1})z \qquad w_3 = \sigma + \sigma^{-1}$$

where $\sigma \in \mathrm{Gal}(\widetilde{L}/\mathbb{Q}_3)$ is a 3-cycle and $z \in L - \mathbb{Q}_3$, $z^2 \in \mathbb{Q}_3$.

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$$\begin{array}{c|ccc} & 1 & \alpha & \alpha^2 \\ \hline w_1 & 1 & \alpha & \alpha^2 \\ w_2 & 0 & 3+9\alpha+2\alpha^2 & -3-30\alpha-9\alpha^2 \\ w_3 & 2 & -3-\alpha & 9-\alpha^2 \end{array},$$

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$$\begin{aligned} h \cdot x = {} & [x_1(h_1 + 2h_3) + x_2(3h_2 - 3h_3) + x_3(-3h_2 + 9h_3)] \\ & + [x_2(h_1 + 9h_2 - h_3) + x_3(-30h_2)]\,\alpha \\ & + [x_2(2h_2) + x_3(h_1 - 9h_2 - h_3)]\,\alpha^2. \end{aligned}$$

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$h \in \mathfrak{A}_H$ if and only if

$$h_1 + 2h_3,$$

$$3h_2 - 3h_3,\ h_1 + 9h_2 - h_3,\ 2h_2,$$

$$-3h_2 + 9h_3,\ -30h_2,\ h_1 - 9h_2 - h_3$$

are 3-adic integers.

$h \in \mathfrak{A}_H$ if and only if

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -3 \\ 1 & 9 & -1 \\ 0 & 2 & 0 \\ 0 & -3 & 9 \\ 0 & -30 & 0 \\ 1 & -9 & -1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \in \mathbb{Z}_3^9$$

$h \in \mathfrak{A}_H$ if and only if

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$\Longrightarrow \{w_1, w_2, \frac{-2w_1-3w_2+w_3}{6}\}$ $\mathbb{Z}_3$-basis of $\mathfrak{A}_H$.

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For $1\leq j\leq n$, set

$$M_j(H,L)\colon=\begin{pmatrix} | & | & \cdots & | \\ (w_1\cdot\gamma_j)_B & (w_2\cdot\gamma_j)_B & \cdots & (w_n\cdot\gamma_j)_B \\ | & | & \cdots & | \end{pmatrix}\in\mathcal{M}_n(K),$$

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$$M_j(H,L) := \begin{pmatrix} | & | & \cdots & | \\ (w_1\cdot\gamma_j)_B & (w_2\cdot\gamma_j)_B & \cdots & (w_n\cdot\gamma_j)_B \\ | & | & \cdots & | \end{pmatrix} \in \mathcal{M}_n(K),$$

Definition

*The **matrix of the action** of H over L is defined as*

$$M(H,L) = \left(\begin{array}{c} M_1(H,L) \\ \hline \cdots \\ \hline M_n(H,L) \end{array}\right) \in \mathcal{M}_{n^2\times n}(K).$$

Example

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$$M_1(H,L) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_2(H,L) = \begin{pmatrix} 0 & 27 & -3 \\ 1 & 81 & -1 \\ 0 & 18 & 0 \end{pmatrix}$$

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Proposition

Suppose that $B = \{\gamma_j\}_{j=1}^n$ *is an* $\mathcal{O}_K$*-basis of* $\mathcal{O}_L$. *Given* $h \in H$,

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Definition

*A **reduced matrix** of $M(H,L)$ is a matrix D such that there is some unimodular matrix $U \in \mathcal{M}_n(\mathcal{O}_K)$ such that*

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Equivalently, if

$$M(H,L) = dM,\ d \in K,\ M \in \mathcal{M}_n(\mathcal{O}_K),$$

then $D = d\Phi$ with $UM = \left(\frac{\Phi}{O}\right)$

Proposition

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Let D be a reduced matrix of $M(H,L)$. Given $h \in H$,

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Theorem (G., Rio)

Let D be a reduced matrix of $M(H,L)$ and call $D^{-1} = (d_{ij})_{i,j=1}^n$. The elements

$$v_i = \sum_{l=1}^n d_{li} w_l,\ 1 \leq i \leq n$$

form an $\mathcal{O}_K$-basis of $\mathfrak{A}_H$.

Example

In the motivating example:

- $D = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{pmatrix}$ is a reduced matrix of $M(H, L)$.
- The inverse is $D^{-1} = \frac{1}{6} \begin{pmatrix} 6 & 0 & -2 \\ 0 & 6 & -3 \\ 0 & 0 & 1 \end{pmatrix}$.
- $\mathfrak{A}_H$ has a basis formed by

$$v_1 = w_1 \qquad v_2 = w_2 \qquad v_3 = \frac{-2w_1 - 3w_2 + w_3}{6}$$

L/K H-Galois extension of p-adic fields.

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Reduction method

W K-basis of H, B $\mathcal{O}_K$-basis of $\mathcal{O}_L$.

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4. Compute the inverse of $D = d\Phi$. Its columns form an $\mathcal{O}_K$-basis of $\mathfrak{A}_H$.

Table of contents

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If $N_1 \leq S_r$ gives L_1/K a H-G structure and $N_2 \leq S_s$ gives L_2/K a H-G structure, then $N := N_1 \times N_2 \leq S_n$ gives L/K a H-G structure.

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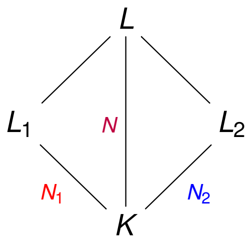

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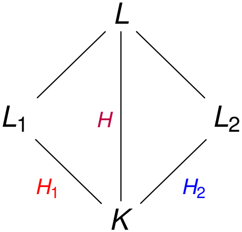

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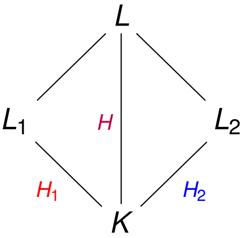

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Proposition (G., Rio)

The induced Hopf Galois structures of L/K are those of the form

$$H = H_1 \otimes_K H_2,$$

where H_1 is a Hopf Galois structure of L_1/K and H_2 is a Hopf Galois structure of L_2/K.

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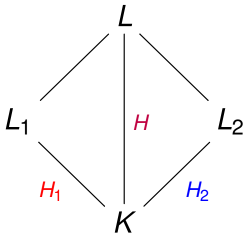

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Definition

The Kronecker product of two matrices $A = (a_{ij})$ and B is the matrix defined by blocks as

$$A \otimes B = (a_{ij}B).$$

Definition

We say that a basis B of L is induced if $M(H, L_B)$ and $M(H_1, L_1) \otimes M(H_2, L_2)$ are integrally equivalent.

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The same holds when L_1/K and L_2/K are arithmetically disjoint.

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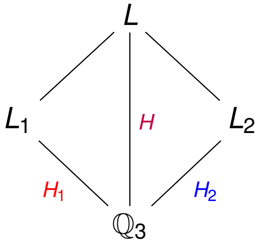

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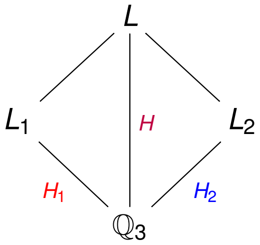

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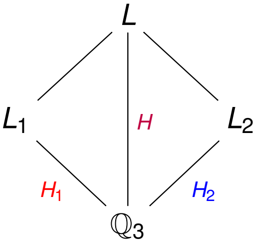

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- $x^3 + 12$
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1. If $f(x) = x^3 + a$, $a \in \{3, 12, 21\}$, then $L/\mathbb{Q}_3$ has an integral induced basis and $\mathfrak{A}_H = \mathfrak{A}_{H_1} \otimes_{\mathbb{Z}_3} \mathfrak{A}_{H_2}$.

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3. If $f(x) = x^3 + ax + 3$, $a \in \{3, 6\}$, then $\mathfrak{A}_H \neq \mathfrak{A}_{H_1} \otimes_{\mathbb{Z}_3} \mathfrak{A}_{H_2}$ (it is not even a tensor product).

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Thank you for your attention