There are no Banach function spaces, only weighted L^2

SIMBa

Seminari Informal
de Matemàtiques de Barcelona

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June 23, 2020

Abstract

An important property of the A_p weights is the extrapolation theorem of Rubio de Francia. It was announced in 1982[1] and given with a detailed proof in 1984[2], both by J.L. Rubio de Francia. In its original version, reads as follows: if T is a sublinear operator which satisfies the strong type boundedness

$$T : L^2(v) \to L^2(v)$$

for every weight $v \in A_2$ with constant only depending on v, then for $1 < p < \infty$,

$$T : L^p(v) \to L^p(v)$$

is bounded for every $v \in A_p$, with constant depending only on v. We will make a review of some of the different versions over Banach function spaces that have been appeared since then.

[1] J.L Rubio de Francia: Factorization and extrapolation of weights. *Bull. Amer. Math. Soc.* **7** (1982), 393–395.

[2] J.L Rubio de Francia: Factorization theory and A_p weights. *Amer. J. Math.* **106** (1984), no. 3, 533–547.

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- if $0 \leq f_n \nearrow f$ μ-a.e. $\Rightarrow \rho(f_n) \nearrow \rho(f)$;
- if E is such that $\mu(E) < \infty \Rightarrow \begin{cases} \rho(\chi_E) < \infty, \text{ and} \\ \int_E f\, d\mu \leq C_E \rho(f) \end{cases}$

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The collection

$$\mathbb{X} = \mathbb{X}(\rho) := \{f \in \mathcal{M} : \|f\|_{\mathbb{X}} := \rho(|f|) < \infty\}$$

is called a Banach function space.

Banach Function Spaces

Definition 2

The associate space of a Banach function space $\mathbb{X}$ is

$$\mathbb{X}' = \{f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{\mathbb{X}'} < \infty\}$$

where

$$\|f\|_{\mathbb{X}'} = \sup_{g \in \mathbb{X}} \frac{1}{\|g\|_{\mathbb{X}}} \int_{\mathbb{R}^n} |f(x)g(x)|\, dx, \qquad f \in \mathcal{M}(\mathbb{R}^n).$$

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Definition 3

We say that

$$T : \mathbb{X} \to \mathbb{X}$$

is bounded if there exists $C > 0$ such that

$$\|Tf\|_{\mathbb{X}} \leq C \|f\|_{\mathbb{X}}.$$

Examples of Banach Function Spaces

Definition 4 (Weighted Lebesgue Spaces)

Let v be a weight, i.e., $v > 0$ and for every compact set $K \subseteq \mathbb{R}^n$,

$$\int_K v(x)\, dx < \infty.$$

(i.e., $v \in L^1_{\text{loc}}(\mathbb{R}^n)$).

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$$\|f\|_{L^p(v)} := \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx \right)^{1/p} < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(v)} := \text{ess sup}|f| < \infty.$$

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Example 5

If $n = 1$ and $v = 1$, $t^{-1/p} \in L^2(0,1)$ whenever $p > 2$.

Proposition 6

For $1 < p < \infty$*, the associate space of* $L^p(v)$ *is* $(L^p(v))' = L^{p'}(v^{1-p'})$ *where* $1 < p' < \infty$ *is the conjugate exponent of* p*, i.e.*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Examples of Banach Function Spaces

Proposition 6

For $1 < p < \infty$, the associate space of $L^p(v)$ is $(L^p(v))' = L^{p'}(v^{1-p'})$ where $1 < p' < \infty$ is the conjugate exponent of p, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Example 7

$(L^2(v))' = L^2(v^{-1})$. In particular, $(L^2(\mathbb{R}^n))' = L^2(\mathbb{R}^n)$.

Examples of Banach Function Spaces

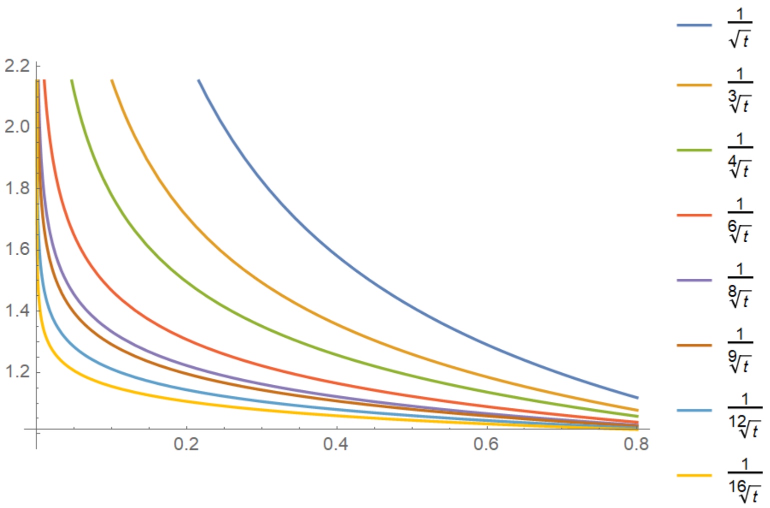

Figure 1: Examples of functions in $L^2(0,1)$ except for the limit function $\frac{1}{\sqrt{t}}$.

Examples of Banach Function Spaces

Definition 8 (Weighted Lorentz Spaces)

For v a weight and $1 < p < \infty$, $1 \leq q < \infty$,

$$f \in L^{p,q}(\mathbb{R}^n, v) \text{ if } \|f\|_{L^{p,q}(v)} := \left(p \int_0^\infty y^{q-1} \lambda_f^v(y)^{q/p}\, dy \right)^{1/q} < \infty,$$

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Remark 9

- $L^{p,p}(v) = L^p(v)$, $1 < p \leq \infty$.
- $L^{p,1}(v) \hookrightarrow L^p(v) \hookrightarrow L^{p,\infty}(v)$.

Examples of Banach Function Spaces

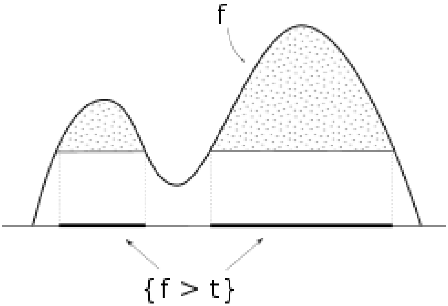

Figure 2: Level set of the function f.

Examples of Banach Function Spaces

Given $0 < p < \infty$ and w a weight: $w \in B_p = B_p(\mathbb{R}^+)$ if exists $C > 0$ such that

$$\|w\|_{B_p} := t^p \int_t^\infty \frac{w(r)}{r^p}\, dr \leq C \int_0^t w(r)\, dr < \infty, \qquad \forall t > 0.$$

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Definition 10 (Classical Weighted Lorentz Spaces)

For $0 < p < \infty$ and $w \in B_p$,

$$f \in \Lambda^p(\mathbb{R}_+, w) \text{ if } \|f\|_{\Lambda^p(w)} := \left(\int_0^\infty f^*(t)^p w(t)\,dt\right)^{1/p} < \infty,$$

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Remark 11

If $w = 1$ then $\Lambda^p(w) = L^p$ and $\Lambda^{p,\infty}(w) = L^{p,\infty}$.

Examples of Banach Function Spaces

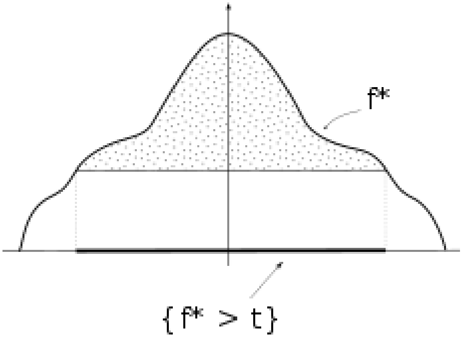

Figure 3: Symmetric decreasing rearrangement of the function f.

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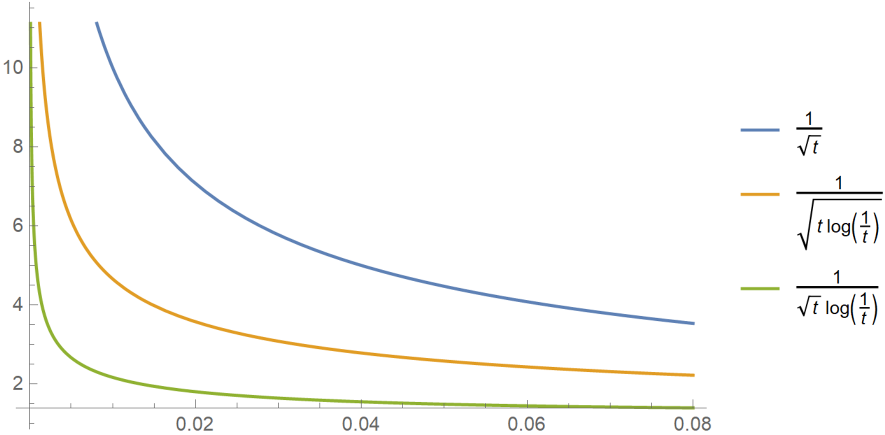

Figure 4: Examples of functions in $L^{2,1}(0,1)$ and $L^{2,\infty}(0,1)$.

Hardy-Littlewood Maximal Operator

Definition 12

Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The Hardy-Littlewood maximal function of f is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \qquad \forall x \in \mathbb{R}^n.$$

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In particular, for $E = (a, b) \subseteq \mathbb{R}$,

$$M\chi_E(t) = \sup_{I \ni t} \frac{|E \cap I|}{|I|} = \begin{cases} \frac{b-a}{b-t}, & t \leq a, \\ 1, & a < t < b, \\ \frac{b-a}{t-a}, & t \geq b. \end{cases}$$

Hardy-Littlewood Maximal Operator

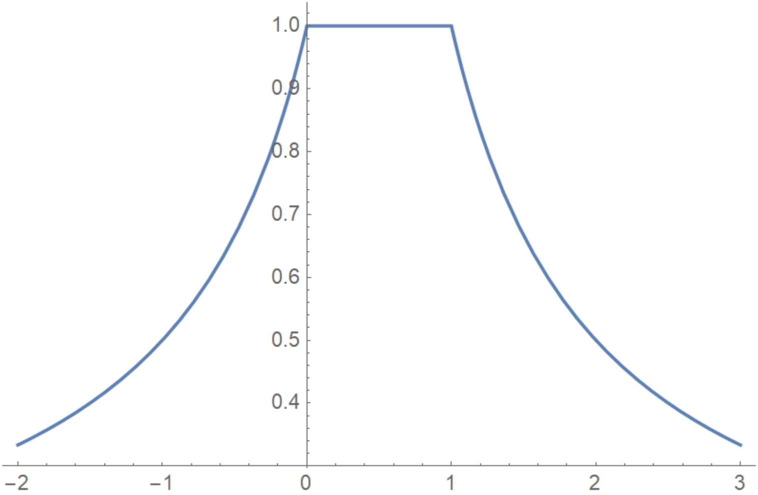

Figure 5: The Hardy-Littlewood maximal function of $\chi_{(0,1)}$.

B. Muckenhoupt[3] characterized the boundedness of

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$1 < p < \infty$,

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Hardy-Littlewood Maximal Operator and A_p weights

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Definition 14

We say that a weight v is in the A_p-class if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q v \right) \left(\frac{1}{|Q|} \int_Q v^{\frac{1}{1-p}} \right)^{p-1} < +\infty.$$

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A_p weights

Proposition 15

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In particular, since $(L^p(v))' = L^{p'}(v^{1-p'})$,

$$M : L^p(v) \to L^p(v) \iff M : (L^p(v))' \iff (L^p(v))'.$$

A_p weights

Lemma 16

$v(x) := |x|^{\alpha}$ ($x \in \mathbb{R}^n$) is an A_p weight for $-n < \alpha < n(p-1)$ (also for $\alpha = 0$ if $p = 1$).

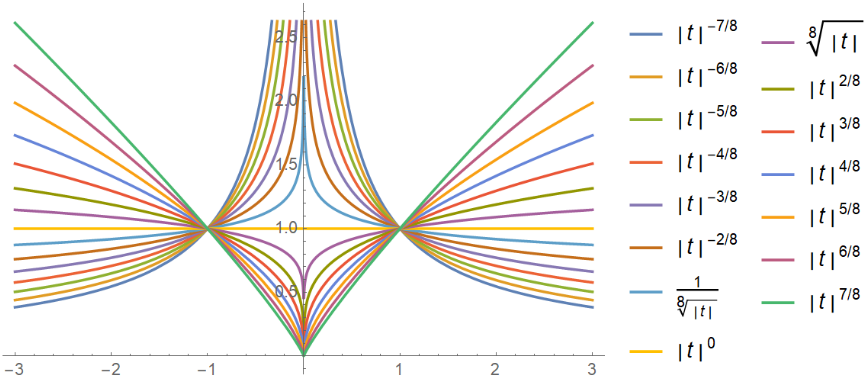

Figure 6: Examples of A_2 weights for $n = 1$.

Starting point

Assume that

$$T : L^2(v) \to L^2(v)$$

is bounded for every $v \in A_2$. Then,

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is bounded for $1 < p < \infty$ and $v \in A_p$

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In the proof it is needed that

$$M : L^p(v) \to L^p(v) \qquad \text{and} \qquad M : (L^p(v))' \Longleftrightarrow (L^p(v))'.$$

Goal

Assume that

$$T : L^2(v) \to L^2(v)$$

is bounded for every $v \in A_2$. Then, if $\mathbb{X}$ is a Banach function space, for which conditions in $\mathbb{X}$

$$T : \mathbb{X} \to \mathbb{X}$$

is bounded?

Answer

Assume that

$$T : L^2(v) \to L^2(v)$$

is bounded for every $v \in A_2$. Then, if $\mathbb{X}$ is a Banach function space, such that

$$M : \mathbb{X} \to \mathbb{X} \quad \text{and} \quad M : \mathbb{X}' \to \mathbb{X}'$$

are bounded, then

$$T : \mathbb{X} \to \mathbb{X}$$

is bounded.

Corollary 17 (Rubio De Francia Extrapolation Theorem)

Assume that

$$T : L^2(v) \to L^2(v)$$

is bounded for every $v \in A_2$. *Then,*

$$T : L^p(v) \to L^p(v)$$

is bounded for every $v \in A_p$.

Example 18

- The trivial one: $T = Id$.

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$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy, \qquad \forall x \in \mathbb{R}^n,$$

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  whenever this limit exists almost everywhere.
- Singular integrals (as Calderón-Zygmund opeartors or Rough operators), some multipliers operators (as the Hörmander multipliers or Bochner-Riesz multipliers over the critical index), commutators, sparse operators... the so-called Rubio de Francia operators.

Weighted Lorentz Spaces

Lemma 19

For $1 < p < \infty$ *and* $1 \leq p < \infty$,

$$M : L^{p,q}(v) \to L^{p,q}(v)$$

is bounded if and only if $v \in A_p$.

Weighted Lorentz Spaces

Lemma 19

For $1 < p < \infty$ *and* $1 \leq p < \infty$,

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is bounded if and only if $v \in A_p$.

Lemma 20

For $1 < p < \infty$ *and* $1 \leq p < \infty$,

$$M : (L^{p,q}(v))' \to (L^{p,q}(v))'$$

is bounded if and only if $v \in A_p$.

Weighted Lorentz Spaces

Corollary 21

Assume that

$$T : L^2(v) \to L^2(v)$$

is bounded for every $v \in A_2$. *Then,*

$$T : L^{p,q}(v) \to L^{p,q}(v)$$

is bounded for every $v \in A_p$.

Weighted Classical Lorentz Spaces

Lemma 22

For $0 < p < \infty$,

$$M : \Lambda^p(w) \to \Lambda^p(w)$$

is bounded if and only if $w \in B_p$.

Weighted Classical Lorentz Spaces

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is bounded if and only if $w \in B_p$.

Lemma 23

For $0 < p < \infty$,

$$M : (\Lambda^p(w))' \to (\Lambda^p(w))'$$

is bounded if and only if $w \in B_\infty^*$, *where*

$$w \in B_\infty^* \iff \sup_{t>0} \frac{1}{\int_0^t w(r)\, dr} \int_0^t \frac{1}{r} \left(\int_0^r w(s)\, ds \right) dr < \infty.$$

Corollary 24

Assume that

$$T : L^2(v) \to L^2(v)$$

is bounded for every $v \in A_2$. *Then,*

$$T : \Lambda^p(w) \to \Lambda^p(w)$$

is bounded for every $w \in B_p \cap B_\infty^*$.

Proposition 25

Let $0 < p < \infty$. *Then,*

$$H : \Lambda^p(w) \to \Lambda^p(w)$$

is bounded if and only if $w \in B_p \cap B_\infty^*$.

My work

- Which operators satisfies the hypothesis.

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- $T : \mathbb{X}(v) \to \mathbb{X}(v)$.

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- $T : \mathbb{X}_1 \times \mathbb{X}_2 \to \mathbb{Y}$.
- Which inequalities do we obtain for f^*?
- $T : \mathbb{X} \to \mathbb{X}$ such that $\mathbb{Y} = (\mathbb{X})^r$ Banach (e.g., $L^p = (L^1)^p$).
- $T : \mathbb{X}(v) \to \mathbb{X}(v)$.
- ...

Thank you for your attention SIMBaddicts!