Supercritical diffussion: memes, stonks and theorems

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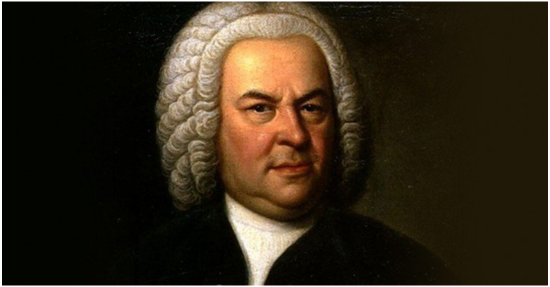
Classical examples



Uniform random walk

Let $X = \{X_0, X_1, X_2, \ldots\}$ with $X_t \in \mathbb{Z}^n$ represent the position of an individual in a lattice at a time t, and suppose X_0 is a random variable that represents the initial probability distribution.

In each step, they move a unit in a random direction, giving the probabilistic law

$$P(X_{t+1} = x) = \sum_{j=1}^{n} \frac{P(X_t = x + e_j) + P(X_t = x - e_j)}{2n}.$$

Notice how the new value of the probability at a point is computed as an average of the probabilities on the neighbours at the previous step.

From discrete to continuous

Consider now the lattice $h\mathbb{Z}^n$, and let the time steps be of size τ. Then,

$$\tau\partial_t P(X_t = x) \simeq P(X_{t+\tau} = x) - P(X_t = x) =$$

$$\sum_{j=1}^{n} \frac{P(X_t = x + he_j) - 2P(X_t = x) + P(X_t = x - he_j)}{2n}$$

$$\simeq \sum_{j=1}^{n} \frac{h^2 \partial_{jj} P(X_t = x)}{n} = \frac{h^2}{n} \Delta P(X_t = x),$$

and therefore choosing $\tau = h^2/n$, we find that $F(x,t) = P(X_t = x)$ solves

$$\partial_t F - \Delta F = 0.$$

Did you just differentiate a function defined in $h\mathbb{Z}^n$?!

Parabolic phenomena

With the proper technical tools, it can be shown that the limit of random walk when $\tau \sim h^2$ is, in an average sense, a solution of the heat (diffusion) equation. The equation is called parabolic because the time scale is proportional to the space squared.

Parabolic equations arise in a variety of settings, namely:

- Thermodynamics (heat equation).
- Fluid mechanics (diffusion).
- Biology (population dynamics).
- Data Science (gaussian processes).
- Probability (random walk).
- Topology (Perelman's proof of Poincaré's conjecture).

Random jumps

Let now $X = \{X_0, X_1, X_2, \ldots\}$ with $X_t \in \mathbb{Z}^n$ as before, but instead of moving only to the neighbouring positions, we allow jumps, with a probability proportional to $d^{-\gamma}$, where d is the distance.

Then, we obtain the following law

$$P(X_{t+1} = x) = c(n, \gamma) \sum_{y \in \mathbb{Z}^n \setminus \{0\}} \frac{P(X_t = x)}{|x - y|^\gamma}.$$

The random walk and this random jump process are particular cases of the more general Lévy process.

Continuous random jumps

Doing the same trick that derived the heat equation from random walks, we can derive that $u(x,t) = P(X_t = x)$ satisfies

$$\partial_t u(x,t) = c(n,\gamma) \int \frac{u(y,t) - u(x,t)}{|x-y|^\gamma} \mathrm{d}y.$$

To ensure the convergence of the integral, we need that $\gamma \in (n, n+2)$, and this time the scaling is $\tau \sim h^{\gamma-2}$.

The fractional Laplacian

Definition: singular integral

Let $s \in (0,1)$. Then, we define

$$(-\Delta)^s u(x) := c(n,s) \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y,$$

where $c(n,s)$ is a dimensional constant.

Notice how $n+2s$ replaces γ in the discrete model.

This definition is related to the following characterization of the Laplacian

$$\begin{aligned} -\Delta u(x) &= \lim_{r \to 0^+} c(n) \int_{B_r(x)} \frac{u(x)-u(y)}{r^{n+2}} \mathrm{d}y \\ &= \lim_{r \to 0^+} c(n) \int_{\mathbb{R}^n} (u(x)-u(y)) \frac{\chi_{B_r(x)}(y)}{r^{n+2}} \mathrm{d}y \end{aligned}$$

Definition: Fourier transform

We can also define the fractional Laplacian via Fourier transform as follows

$$\mathcal{F}((-\Delta)^s u)(\xi) := |\xi|^{2s}\mathcal{F}(u)(\xi).$$

The constant $c(n, s)$ in the definition before is chosen so that the definitions coincide. Notice how when $s = 1$ the definition coincides with the Fourier transform of the Laplacian of u, because

$$\mathcal{F}(-\Delta u)(\xi) = \mathcal{F}(-\sum u_{ii})(\xi) = \sum \xi_i^2 \mathcal{F}(u) = |\xi|^2 \mathcal{F}(u).$$

This definition suggest why we say that $(-\Delta)^{2s}$ does $2s$ derivatives to functions.

Definition: Caffarelli-Silvestre extension

We can also define the fractional Laplacian as the following. Let $u : \mathbb{R}^n \to \mathbb{R}$, and let $v(x, y)$ the solution of

$$\begin{cases} \mathrm{Div}(y^{1-2s}\nabla v) & = & 0 & \text{in} & \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) & = & u(x) & \text{on} & \mathbb{R}^n \end{cases}$$

Then,

$$(-\Delta)^s u(x) = \tilde{c}(n,s) \lim_{y \to 0^+} y^{1-2s} v_y(x,y)$$

for some dimensional constant $\tilde{c}(n,s)$.

The half-Laplacian as an extension

If $s = 1/2$, we recover the important half-Laplacian $(-\Delta)^{1/2}$ as follows. Given u and v as before,

$$\begin{cases} \Delta v = \mathrm{Div}(\nabla v) & = & 0 & \text{in} & \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) & = & u(x) & \text{on} & \mathbb{R}^n \end{cases}$$

Then, v is just the harmonic extension of u to $\mathbb{R}^n \times \mathbb{R}^+$, and the limit simplifies to $(-\Delta)^{1/2}u = v_y(\cdot, 0)$. Hence, we can see the half-Laplacian as a Dirichlet-to-Neumann operator for harmonic functions. If we compute

$(-\Delta)^{1/2}[(-\Delta)^{1/2}u]$, the harmonic extension of $v_y(\cdot,0)$ is just v_y, and hence

$$(-\Delta)^{1/2}[(-\Delta)^{1/2}u] = v_{yy}(\cdot,0) = -\Delta_x v(\cdot,0).$$

The fractional Laplacian: checkpoint

- $(-\Delta)^s$ is a pseudo-differential operator of order $2s$, i.e. it makes $2s$ derivatives.
- $(-\Delta)^s$ is a nonlocal operator, i.e. the value at a point depends on the values of the function everywhere.
- The three definitions are equivalent when they make sense (convergence et al).
- $(-\Delta)^{s_1}(-\Delta)^{s_2}u = (-\Delta)^{s_1+s_2}u$ if all exponents are in $(0, 1)$.

The Dirichlet problem

Given an open domain $\Omega \subset \mathbb{R}^n$ with regular enough boundary, we have existence and uniqueness of solutions for the following problem

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in} \quad \Omega \\ u &= g \quad \text{in} \quad \Omega^c, \end{cases}$$

provided that f and g are regular enough.

Notice that we have to prescribe the boundary condition g in Ω^c and not only in $\partial\Omega$ because of the nonlocality.

The mean value property

Harmonic functions satisfy the mean value property, i.e. if $\Delta u = 0$ in $\mathbb{R}^n$, then, for all $r > 0$,

$$u(x) = \frac{1}{|B_r|} \int_{B_r(x)} u(y)\mathrm{d}y.$$

The fractional counterpart is a weighted mean value property. If

$(-\Delta)^s u = 0$ in $\mathbb{R}^n$, then

$$u(x) = \int_{B_r(x)^c} \frac{c(n,s)r^{2s}}{(|y|^2 - r^2)^s |y|^n} u(y)\mathrm{d}y$$

for all $r > 0$.

The comparison principle

Harmonic functions satisfy also the comparison principle, meaning that solutions cannot cross. More in detail, if Ω is an open domain with regular boundary and

$$\begin{cases} -\Delta u \le -\Delta v & \text{in } \Omega \\ u \le v & \text{on } \partial\Omega \end{cases}$$

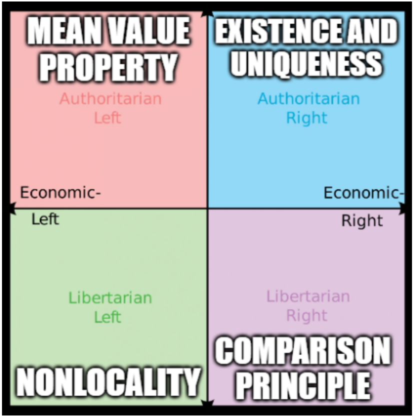
it follows that $u \le v$ in Ω. Analogously, for the fractional Laplacian, if we assume

$$\begin{cases} (-\Delta)^s u \le (-\Delta)^s v & \text{in } \Omega \\ u \le v & \text{on } \Omega^c \end{cases}$$

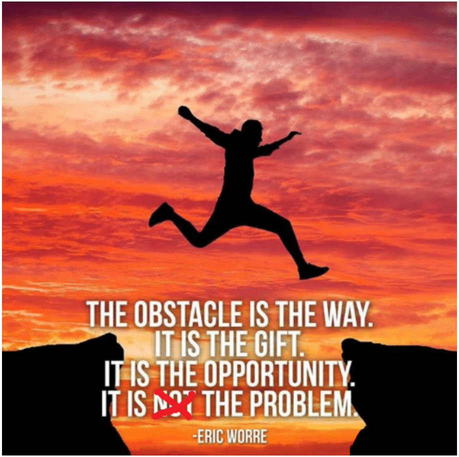
then it follows that $u \le v$ in Ω as well.

Summary

The fractional Laplacian has properties all across the political spectrum.



The parabolic fractional obstacle problem



Getting rich with PDEs

Consider the following optimization problem. Let $x \in \mathbb{R}^n$ be the state of an asset, and assume we have a contract that lets us sell the asset at a price $\varphi(x)$ from now (time 0) until time T. Suppose that the state of the asset follows a Lévy process (*random jump*). What is the best possible strategy of when to sell and what is the maximum expected sale price?

Let $u(x)$ be the maximum expected sale price. It is clear that $u \geq \varphi$, because we can sell at price $\varphi(x)$ now and forget about it.

If we don't sell at time t, then the state x evolves with a continuous random jump law, and then $\partial_t u = -c(-\Delta)^s u$. And we don't sell when $u_t > 0$. Summarizing,

$$\min\{u_t + c(-\Delta)^s, u - \varphi\} = 0.$$

The value of s

- In the classical Black-Scholes model, the diffusion is Gaussian and $s = 1$.
- In the Merton model (70s), the diffusion is nonlocal.
- Empirical studies show $s \simeq 0.7$ for Wall Street stocks in the 80s.
- The diffusion of memes in the internet shows smaller values of s, even lower than $1/2$.

The money's at the boundary

The parabolic fractional obstacle problem can be posed as the following:

$$\begin{cases} \min\{u_t + (-\Delta)^s u, u - \varphi\} & = & 0 & \text{in} & \mathbb{R}^n \times (0, T) \\ u(\cdot, 0) & = & \varphi & \text{on} & \{t = 0\} \end{cases}$$

Notice that u is always above the obstacle, $u \geq \varphi$, and when $u > \varphi$, $u_t = -(-\Delta)^s u$, which corresponds to the fractional heat equation.

The fractional heat equation is well understood. Then, the interesting part is understanding how the solution u detaches from φ and the shape of the free boundary $\partial\{u > \varphi\}$.

Scale invariance

To understand the free boundary $\partial\{u > \varphi\}$, we want to zoom in preserving the equation. Hence, we define

$$u_r(x,t) = u(rx, r^{2s}t),$$

which solves

$$\min\{\partial_t u_r + (-\Delta)^s u_r, u_r - \varphi(rx)\} = 0.$$

Then, the limit as $r \to 0$ is very different depending on if $r > 1/2$, $r = 1/2$ or $r < 1/2$.

Subcritical, critical and supercritical

The spatial scaling is proportional to the time scaling raised to $2s$. Then, when we zoom in

- If $s > 1/2$ (subcritical), time goes much faster to 0 than space, and we obtain a solution of the time-independent problem.
- If $s = 1/2$ (critical), as we zoom in we need to take into account the time derivative and the space nonlocal term (still open).
- If $s < 1/2$ (supercritical), we get $u_t = 0$ as a limit, which is not useful.

Known results

- Time-independent equation [Caffarelli, Ros-Oton, Serra 2007]. The solution and the free boundary are $C^{1,s}$.
- Time-dependent equation [Caffarelli, Figalli 2013]. The solution is $C^{1,s}$ in space and $C^{1,\alpha}$ in time with α explicit.
- Subcritical case [Barrios, Figalli, Ros-Oton 2018]. The free boundary is $C^{1,\alpha}$ at regular points.

The supercritical problem

In the supercritical case, $s < 1/2$ zooming in is not useful to prove stuff. Using the comparison principle, explicit solutions and epsilon-delta, we proved with X. Ros-Oton

- The solution is $C^{1,1}$ (second derivatives in L^∞), which is more regular than the time-independent case.
- The free boundary is $C^{1,\alpha}$ everywhere.

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