Topological models of ∞-groupoids

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- Study topological categories as a model of ∞-categories.
- Prove that Moore path categories are a model of the fundamental ∞-groupoid of a topological space.

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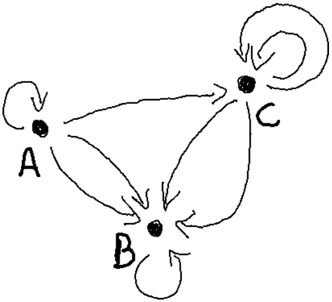

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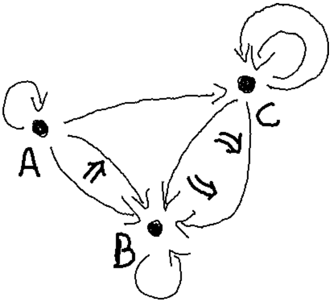

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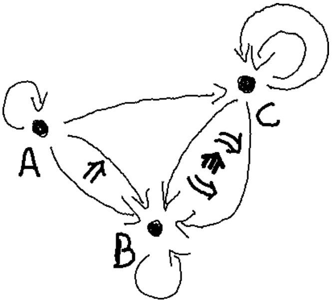

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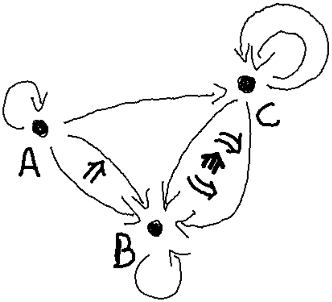

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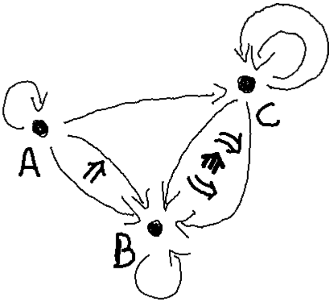

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- For any n-morphism f there is $(n+1)$-morphisms

$$f \circ \mathrm{Id} \Rightarrow f \Leftarrow \mathrm{Id} \circ f$$

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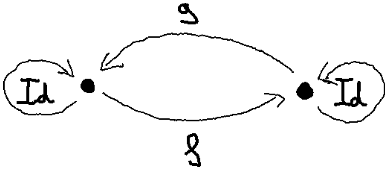

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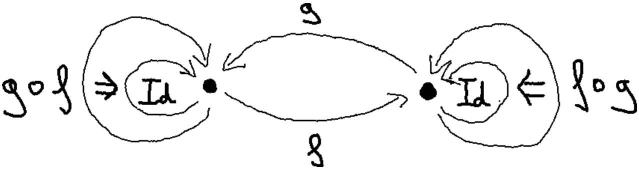

Grothendieck's homotopy hypothesis

Historically, there have been many definitions for ∞-categories, and each one is called a **model**:

- Globular models (Grothendieck, Batanin, Berger, etc.).
- Quasi-categories (Joyal, Lurie).
- Topological categories (Bergner, Lurie, etc.).

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Given a model of ∞-categories and a topological space X, the **fundamental ∞-groupoid** $\Pi_\infty(X)$ is the ∞-groupoid which encodes the homotopical structure of X.

Model categories

A **model category** is a complete and cocomplete category with three distinguished classes of morphisms: **weak equivalences** ($\xrightarrow{\sim}$), **fibrations** ($\twoheadrightarrow$), and **cofibrations** ($\hookrightarrow$); satisfying certain axioms relating them.

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Given a model of ∞-categories, the statement of Grothendieck's homotopy hypothesis is equivalent to finding a zigzag of Quillen equivalences between the categories of topological spaces and ∞-groupoids.

Example: Topological spaces

A map of topological spaces $f : X \to Y$ is a **weak homotopy equivalence** if f induces a bijection between path components and for any $n \in \mathbb{N} \setminus \{0\}$

$$\pi_n(f, x) : \pi_n(X, x) \xrightarrow{\cong} \pi_n(Y, f(x)).$$

A map of topological spaces $p : E \to B$ is a **Serre fibration** if it satisfies the homotopy lifting property with respect to any n-disk:

$$\begin{array}{ccc} \mathbb{D}^n & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ \mathbb{D}^n \times I & \longrightarrow & B \end{array}$$

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Simplicial sets

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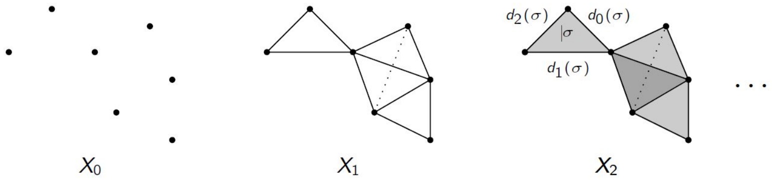
For each $[n]$, a simplicial set X has a set denoted $X[n]$ or X_n. In addition to those sets, X is determined by its **faces** $d_i : X_n \to X_{n-1}$ and **degeneracies** $s_i : X_n \to X_{n+1}$, which satisfy the simplicial identities.

Idea of simplicial sets



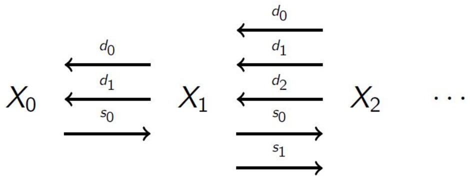


Figure: A simplicial set $X : \Delta^{op} \to \mathbf{Set}$, from nLab wiki.

Standard simplices and horns

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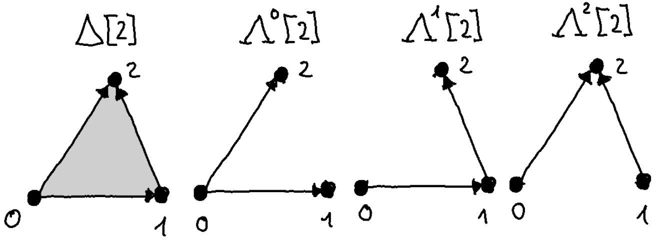

Quasi-categories and Kan complexes

A simplicial set X is called a **quasi-category** if for any $n \geq 1$ and $0 < k < n$, every map $\Lambda^k[n] \to X$ admits an extension $\Delta[n] \to X$ through the canonical inclusion $\Lambda^k[n] \to \Delta[n]$.

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Quasi-categories model ∞-categories with the n-simplices modeling the n-morphisms, and Kan complexes model ∞-groupoids.

Simplicial sets as model category

The category of simplicial sets can be considered the following two model structures:

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Therefore, the homotopy hypothesis for simplicial sets is equivalent to finding a Quillen equivalence between $\mathbf{sSet}_Q$ and the category of topological spaces with the usual model structure.

Formal nerve and realization

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The Q-nerve and Q-realization always form an adjunction

$$|\cdot|_Q : \mathbf{sSet} \rightleftarrows \mathcal{C} : \mathrm{N}^Q .$$

Homotopy hypothesis for simplicial sets

There is a cosimplicial object $\Delta^\bullet$ defined for each $[n] \in \Delta$ as the topological space

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- **Singular simplicial set** $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ is the $\Delta^\bullet$-nerve.
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Those functors form a Quillen equivalence, proving the homotopy hypothesis for simplicial sets:

$$\mathbf{Top} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\leftrightarrows}} \mathbf{sSet}_Q$$

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- For every object $X \in \mathcal{C}$, a morphism $I \to \mathcal{C}(X, X)$ of $\mathcal{M}$ which represents the identity element.

Topological categories

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- For every topological category $\mathcal{C}$, the **homotopy category** $\mathrm{h}\mathcal{C}$ has the same objects as $\mathcal{C}$ and $\mathrm{h}\mathcal{C}(X, Y) = \pi_0(\mathcal{C}(X, Y))$.

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- A topological category $\mathcal{C}$ is an ∞-**groupoid** if h$\mathcal{C}$ is a groupoid.

Homotopy coherent nerve

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There is a cosimplicial object $\Delta^{\Re}$ defined for each $[n] \in \Delta$ as the simplicial category $\Delta^{\Re}[n]$ with:

- $\mathrm{Obj}(\Delta^{\Re}[n]) = [n] = \{0, \ldots, n\}$
- For every $i, j \in \mathrm{Obj}(\Delta^{\Re}[n])$, $\mathrm{Hom}(i, j) = (\Delta[1])^{(j-i-1)}$

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The $\Delta^{\mathfrak{R}}$-nerve is called **homotopy coherent nerve** $\mathrm{N}^{\mathfrak{R}}$ and the $\Delta^{\mathfrak{R}}$-realization is called **simplicial path** $\mathfrak{C}$. There is a Quillen equivalence:

$$\mathbf{sSet}_J \underset{\mathfrak{C}}{\overset{\mathrm{N}^{\mathfrak{R}}}{\rightleftarrows}} \mathbf{sSet\text{-}Cat}$$

Homotopy hypothesis for topological categories

Top **Top-Cat**

Homotopy hypothesis for topological categories

$$\mathbf{Top} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\leftrightarrows}} \mathbf{sSet}_Q \qquad\qquad \mathbf{Top\text{-}Cat}$$

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Homotopy hypothesis for topological categories

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- The composition is defined by

$$\begin{array}{rccc} \circ : & P_{x,y}^M X \times P_{y,z}^M X & \longrightarrow & P_{x,z}^M X \\ & ((f,r),(g,s)) & \longmapsto & (f * g, r+s) \end{array}$$

$$(f * g)(t) = \begin{cases} f(t) & \text{if } 0 \leq t < r \\ g(t-r) & \text{if } t \geq r \end{cases}$$

Classifying space

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There are functorial classifying spaces generalized to include the case of group-like topological monoids, for example the Milgram classifying space.

Classifying space

A **classifying space** $\mathrm{B}(G)$ of a topological group G is a quotient of a weakly contractible space $\mathrm{E}(G)$ by a proper free action of G.

There are functorial classifying spaces generalized to include the case of group-like topological monoids, for example the Milgram classifying space.

Let $\Omega_x^M(X)$ be the group-like topological monoid defined as $P_{x,x}^M X$. The **delooping** functor $\mathbb{D} : \mathbf{tMon} \to \mathbf{Top\text{-}Cat}_0$ sends a topological monoid M to the topological category with one object $*$ and $\mathrm{Hom}(*, *) = M$.

The fundamental ∞-groupoid as a Moore path category

Theorem

Let (X, x) be a path-connected well-pointed topological space. The topological space $|\,\mathsf{N}^{\mathfrak{R}}(\mathsf{Sing}_e(\mathbb{D}\,\Omega^M_x(X)))|$ is a classifying space for $\Omega^M_x(X)$ and, as a consequence,

$$|\,\mathsf{N}^{\mathfrak{R}}(\mathsf{Sing}_e(\mathbb{D}\,\Omega^M X))| \simeq X.$$

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Hence, the ∞-groupoid $\Pi_\infty^M(X)$ is weakly homotopy equivalent to the ∞-groupoid $(|\cdot|_e \circ \mathfrak{C} \circ k_! \circ \mathrm{Sing})(X)$.

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Topological models of ∞-groupoids

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