# DEGREE OF SYMMETRY OF MANIFOLDS AND THE TORAL RANK CONJECTURE

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ABSTRACT. This notes are an expanded version of talk given at SIMBA on 06/04/2002 with the same title. They are a brief introduction to the theory of topological transformation groups, a field of mathematics which studies the symmetries of topological spaces. The notes are divided in 3 parts. In the first we introduce the basic definition and properties of group actions. Thereafter, we explain the slice theorem, one important geometric tool to study group actions. Finally, we introduce the problem of finding the degree of symmetry of a manifold and the toral rank conjecture.

## 1. Basic definitions and properties

The aim of the theory of topological transformation groups is to describe and study the symmetry of topological spaces. One of the fundamental questions of this theory is given a compact group $G$ and a topological space $X$, is there an (effective) action of $G$ on $X$?

This questions is really hard to answer, so it is usual to study more approachable variations of the question. For example, given a topological space $X$, how "big" a group $G$ acting on $X$ can be? The toral rank conjecture ask this questions for a special kind of toral actions. If it was true, it would give an upper bound on the dimension $r$ of a torus $T^r$ which can act almost-freely on $X$ in terms of the cohomology of $X$.

**Notation 1.1.** *In this notes, $X$ will always denote a connected Hausdorff topological space and $M$ a connected topological manifold. We will state explicitly when the manifold is smooth.*

We start by defining the basic objects of this theory:

**Definition 1.2.** *A topological group $G$ is a Hausdorff topological space together with a continuous group action $G \times G \longrightarrow G$ (denoted by $(g,h) \mapsto gh$) which makes $G$ into a group and such that the map $G \longrightarrow G$ such that $g \mapsto g^{-1}$ is continuous. If $G$ is a smooth manifold and the product and inverse maps are smooth we say that $G$ is a Lie group.*

**Definition 1.3.** *A continuous group action of $G$ on a topological space $X$ is a continuous map $\Phi : G \times X \longrightarrow X$ such that:*

(1) $\Phi(gh, x) = \Phi(g, \Phi(h, x))$ *for all $g, h \in G$ and $x \in X$.*

(2) $\Phi(e, x) = x$ *for all $x \in X$, where $e$ is the identity element of $G$.*

We can define a smooth action analogously, by replacing $X$ for a smooth manifold, $G$ for a Lie group and asking $\Phi$ to be smooth.

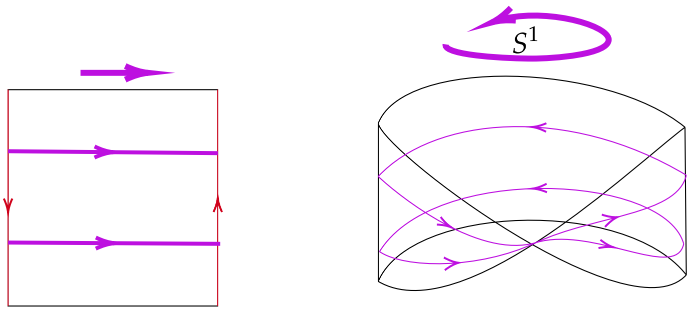
**Notation 1.4.** *We write $gx = \Phi(g, x)$ and $\phi_g : X \longrightarrow X$ for the map such that $\phi_g(x) = gx$. The space $X$ is called (left) $G$-space.*

**Remark 1.5.** *Note that we have a group morphism $\phi : G \longrightarrow \text{Homeo}(X)$ such that $\phi(g) = \phi_g$. If the action is smooth, we have a group morphism $\phi : G \longrightarrow \text{Diff}(M)$.*

**Example 1.6.** (1) *The group $\text{GL}(n, \mathbb{R})$ acts linearly on $\mathbb{R}^n$. This action induces a linear action of the subgroup $\text{O}(n)$ on $S^{n-1}$.*

(2) *The group $S^1$ acts by rotations around an axis on $S^2$.*

(3) *Let $M = \mathbb{R} \times \mathbb{R}/\sim$ with the equivalence relation $(x, y) \sim (x + n, (-1)^n y)$ for every $b \in \mathbb{Z}$ be the the Möbius strip. We have a $S^1$ action of $M$ such that $e^{2\pi i t}[x, y] = [x + 2t, y]$.*




**Definition 1.7.** *Let $G$ be a group and $X$ and $Y$ $G$-spaces. A continuous map $f : X \longrightarrow Y$ is said to be equivariant if it fulfils that $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.*

**Definition 1.8.** *Let $G$ be a group acting on $X$ and let $x \in X$, then:*

- *The stabilizer of $x$ is $\text{Stab}(x) = G_x = \{g \in G : gx = x\}$. Note that $G_x$ is a closed subgroup of $G$. It can also be called isotropy subgroup of $x$.*
- *The orbit of $x$ is $\mathcal{O}(x) = Gx = \{gx : g \in G\}$. The space of all orbits will be denoted by $X/G$.*

Note that we can define an equivalence relation in $X$ such that $x \sim y$ if and only if $G(x) = G(y)$ (that is, there exists $g \in G$ such that $y = gx$). Thus, we have that $X/G$ is a topological space with the topology induced by the map $\pi : X \longrightarrow X/G$ that sends each point to its orbit.

Group actions can have the next properties:

**Definition 1.9.** *Let $G$ be a group acting on $X$. Then:*

(1) *The action is said to be effective if $\bigcap_{x \in X} G_x = \{e\}$ (or alternatively, $\phi : G \longrightarrow \text{Homeo}(X)$ is injective).*

(2) *The action is almost free if $G_x$ is finite for all $x \in X$, and it is free if $G_x = \{e\}$ for all $x \in X$.*

(3) *The action is transitive if for all $x, y \in X$ there exists $g \in G$ such that $y = gx$ (or alternatively, $Gx = X$ for a $x \in X$).*

Until now, we have not introduced the compactness assumption on $G$, but this condition is necessary to have really strong properties which non-compact group actions do not have. The next proposition summarises some of the first properties of compact group actions (see [2, I.3.1.]).

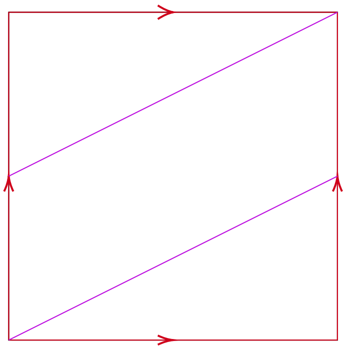
**Proposition 1.10.** *Let $G$ be a compact group acting on $X$, then:*

(1) *The map $\Phi : G \times X \longrightarrow X$ is a closed map.*

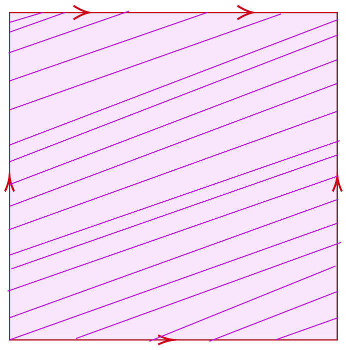
(2) *The orbit space $X/G$ is Hausdorff. $X$ is compact if and only if $X/G$ is compact.*

(3) *The quotient map $\pi : X \longrightarrow X/G$ is closed and proper.*

**Remark 1.11.** *(Non-compact vs. compact group actions): We define an action $\mathbb{R}$ of $T^2$ such that $t(e^{2\pi ix}, e^{2\pi iy}) = (e^{2\pi i(x+t)}, e^{2\pi i(y+at)})$, with $a \in R$. The action behaves differently depending on if $a$ is rational or irrational. If $a = r/s$ is rational, then the action is not effective, since the element $s \in \mathbb{R}$ acts like the identity in $T^2$. In consequence, this action induces an effective action of $S^1 = \mathbb{R}/(s\mathbb{Z})$ on $T^2$. On the other hand, if $a$ is irrational, the action of $\mathbb{R}$ on $T^2$ is free, but the orbit for any $(x, y) \in T^2$ is dense in $T^2$. In particular $T^2/\mathbb{R}$ is homeomorphic to $S^1$ with the trivial topology. In particular, $T^2/\mathbb{R}$ is not Hausdorff.*



$a \in \mathbb{Q}$



$a \in \mathbb{R} \setminus \mathbb{Q}$

*(Continuous vs. smooth group actions): In general continuos group actions are more difficult to study than smooth group actions. Let $G$ be a compact Lie group acting on a manifold $M$, we denote by $M^G = \{x \in M : gx = x, \forall g \in G\}$ the set of fixed points. If the actions is smooth then $M^G$ is a compact submanifold. On the other hand, if the action is continuous, $M^G$ is not necessarily a topological manifold. For example, Bing showed that there is an involution (a $\mathbb{Z}_2$ action) in $S^3$ whose fixed point set is an Alexander horned sphere.*

**Remark 1.12.** *An extra condition that we can impose to non-compact group actions is the action to be proper, which means that $G \times M \longrightarrow M \times M$ such that $(g, x) \mapsto (gx, x)$ is proper (the preimage of compact sets are compact). A lot of the statement for compact group actions are also valid on the setting of non-compact proper group actions.*

Given a $G$-space $X$, we want to understand and classify the different orbits that this action can produce. We define the evaluation map at $x \in X$ to be the continuous map $\text{ev}_x : G \longrightarrow X$ such that $\text{ev}_x(g) = gx$.

However, note that we have a bijective map $\alpha_x : G/G_x \longrightarrow Gx$ such that $\alpha_x(gG_x) = gx$. Moreover if $G$ is compact then $\alpha_x$ is a continuous map of a compact to a Hausdorff space which implies that $\alpha_x$ is closed. Therefore:

**Lemma 1.13.** *(see* [2, I.4.1.]*) If $G$ is compact, then $\alpha_x$ is a homeomorphism.*

Thus every orbit is homeomorphic to a coset space $G/H$ where $H$ is a closed subgroup of $G$. In consequence, we need to study coset spaces $G/H$.

**Lemma 1.14.** *(see [2, I.4.2.]) Let $G$ be a compact group and $H$ and $K$ closed subgroups. Then:*

(1) *There exists an equivariant map $G/H \longrightarrow G/K$ if and only if there exists $a \in G$ such that $aHa^{-1} \subset K$.*

(2) *Let $H$ and $K$ be closed subgroups of $G$ and assume that there exists $a \in G$ such that $aHa^{-1} \subset K$, then the map $R_a^{K,H} : G/H \longrightarrow G/K$ such that $R_a^{K,H}(gH) = ga^{-1}K$ is well-defined and equivariant. Any equivariant map $G/H \longrightarrow G/K$ is of this form.*

*Proof.* Let $f : G/H \longrightarrow G/K$ be an equivariant map. We know that $f(eH) = a^{-1}K$ for some $a \in G$ and since $f$ is equivariant, we have that $f(gH) = ga^{-1}K$ for all $g \in G$. In particular, given $h \in H$ we have that $f(hH) = ha^{-1}K = a^{-1}K$. This implies that $aha^{-1} \in K$ for all $h \in H$. Conversely, if there exists $a \in G$ such that $aHa^{-1} \subset K$, then the map $f : G/H \longrightarrow G/K$ such that $f(gH) = ga^{-1}K$ is well defined and equivariant. □

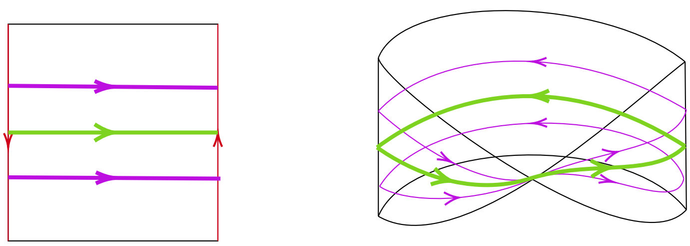
**Remark 1.15.** *Since $G$ is compact, we have that $aHa^{-1} \subset H$ implies that $aHa^{-1} = H$.*

**Corollary 1.16.** *If there exists equivariant maps $G/H \longrightarrow G/K$ and $G/K \longrightarrow G/H$ then $H$ and $K$ are conjugate and these maps are homeomorphisms.*

Let $\mathcal{G}$ be the category of $G$-orbits (where the objects are homogeneous spaces $G/H$ and the morphisms are $G$-equivariant maps between them). We define an equivalence relation such that for $X, Y \in \mathcal{G}$ we have that $X \sim Y$ if and only if there exists an equivariant homeomorphism $f : X \longrightarrow Y$. The equivalence classes under this relation are called orbits types and we will denote the equivalence class of $X$ by $\text{type}(X)$. By lemma 1.13, we can always choose a homogeneous space $G/H$ as a representative of each orbit type. Moreover, we can define a partial order relation in $\mathcal{G}/\sim$ such that $\text{type}(X) \geq \text{type}(Y)$ if and only if there exists an equivariant map $F : X \longrightarrow Y$. Note that that we have a minimum and maximum elements corresponding to $\text{type}(G/G)$ and $\text{type}(G)$ respectively.

Analogously, given an orbit $X$ which is equivalent to $G/H$, we define the isotropy type of $X$ to be the conjugacy class of $H$ in $G$, which we will denote by $(H)$. In the set of conjugacy classes of closed subgroups of $G$ there is a partial order such that $(H) \leq (K)$ if and only if $H$ is conjugate to a subgroup of $K$. Then, the lemma 1.14 implies that we have an anti-isomorphism of partially ordered sets given by $\text{type}(G/H) \mapsto (H)$.

**Example 1.17.** *In the case of the action of $S^1$ on the Möbius strip $M$ from the first example there are two orbit types, $\text{type}(S^1)$ and $\text{type}(S^1/\mathbb{Z}_2)$. They correspond respectively to the purple and green orbit from the picture below:*



Orbit and isotropy types are a good tool to describe the structure of a manifold with a $G$-action. Thus, given a smooth manifold $M$, we will denote by $M_{(H)}$ the union of all orbits with isotropy type $(H)$.

**Theorem 1.18.** *Let $G$ be a compact Lie group acting on a connected smooth manifold $M$, then there exists a maximum orbit type $G/H$ on $M$. The set $M_{(H)}$ is connected and dense in $M$ and $M_{(H)}$ is connected in $M/G$.*

The proof can be found in [5, Theorem 4.27.] and in the more general case of linearly smooth actions in [2, IV.3.1.].

**Theorem 1.19.** *Let $G$ be a compact Lie group acting on a compact connected smooth manifold $M$, then there are finitely many orbit types.*

See [5, Theorem 4.23] for a proof of the theorem.

## 2. The slice theorem

The slice theorem is a tool to study the structure of the group actions, which enables us to deduce properties of $M/G$ and the quotient map $\pi : M \longrightarrow M/G$. We start by recalling some definitions.

**Definition 2.1.** *Let $F$ be a right $G$-space and let $E$ and $B$ be topological Hausdorff spaces. A fiber bundle over $B$ (base space) with total space $E$, fiber $F$ and structure group $G$ is a map $p : E \longrightarrow B$ together with a collection $\mathcal{U}$ of $(U, \phi)$ pairs of open subsets of $B$ and homeomorphism $\phi : F \times U \longrightarrow p^{-1}(U)$ (called local trivialization charts) such that:*

1 *The diagram*

$$\begin{array}{ccc} F \times U & \xrightarrow{\phi_i} & p^{-1}(U_i) \\ & \searrow & \big\downarrow p \\ & & U_i \end{array}$$

*commutes for every $(U, \phi) \in \mathcal{U}$.*

(2) *The collection of open sets $\{U : (U, \phi) \in \mathcal{U}\}$ is an open covering of $B$.*

(3) *If $(U, \phi) \in \mathcal{U}$ and $V \subset U$, then $(V, \phi_{|F \times V}) \in \mathcal{U}$.*

(4) *If $(U, \phi)$ and $(U, \psi)$ are in $\mathcal{U}$, there exists a continuous map $\theta : U \longrightarrow G$ such that $\psi(f, x) = \phi(f\theta(x), x)$ for all $f \in F$ and $u \in U$.*

(5) *The set $\mathcal{U}$ is maximal among collections satisfying the preceding conditions.*

If $p : E \longrightarrow B$ is a $G$-principal bundle, then there exists a free action of $G$ on $E$ such that $p$ induces a homeomorphism $B \longrightarrow E/G$. Thus, every principal $G$-bundle induces a free action of $G$. The slice theorem will help to prove the converse statement in the case where $G$ is a compact Lie group acting smoothly on a smooth manifold $E$. In order to explain the slice theorem, we first need to introduce the concept of associated bundle, tube an slice.

Let $F$ be a left $G$-space and $p : E \longrightarrow B$ a $G$ principal bundle. The associated bundle construction is a way to construct fiber bundles with fiber $F$ by attaching $F$ to each fiber of a principal $G$-bundle. Since, we have a free right action of $G$ on $E$, we can construct a free left action on $E \times F$ which fulfils that $g(e, f) = (eg^{-1}, gf)$. The quotient space space $(E \times F)/G$ is denoted by $E \times_G F$ and its elements by $[e, f]$. Then, we have a map $\pi : E \times_G F \longrightarrow B$ such that $\pi[e, f] = p(e)$. Then (see [2, II.2.4.]):

**Lemma 2.2.** *The map $\pi : E \times_G F \longrightarrow B$ is a fiber bundle with fiber $F$ and structure group $G$.*

If $(U, \phi)$ is a local trivialization of the principal $G$-bundle, then the local trivializations of the associated bundle are given by $(U, \phi')$, where $\phi'(u, f) = [\phi(u, e), f]$.

**Lemma 2.3.** *Let $G$ be a Lie group and $H$ a close subgroup of $G$. If $M$ is an effective $H$-manifold then $\pi : G \times_H M \longrightarrow G/H$ is and smooth fiber bundle and $G \times_H M$ is a smooth manifold.*

Assume that we have a smooth action of a compact Lie group $G$ on a manifold $M$, then the orbit $Gx$ is a submanifold of $M$, then we can use the tubular neighbourhood theorem to find an open neighbourhood $T$ of $Gx$ with good properties. However, in general $T$ will not be invariant by $G$. The concept of a tube and slice are defined in order to construct neighbourhoods of $Gx$ which behaves well with respect the group action.

**Definition 2.4.** *Let $G$ be a compact group and $X$ be a $G$-space and $Gx$ an orbit of isotropy type $(H)$. A tube about $Gx$ is a continuous map $\phi : G \times_H A \longrightarrow X$ which is a homeomorphism onto an open neighbourhood of $Gx$ in $X$, where $A$ is a $H$-space.*

*A slice at $x$ is a subspace $S \subset X$ containing $x$ which fulfils that $G_x(S) = S$ and the map $G \times_{G_x} S \longrightarrow X$ such that $[g, s] \mapsto gs$ is a tube about $Gx$.*

These next two characterizations of a slice will be useful to prove the slice theorem:

**Proposition 2.5.** *(see* [2, II.4.2.]*) Let $X$ be a $G$-space, let $x \in S \subset X$ and put $H = G_x$. Then the following statements are equivalent:*

(1) *There exists a tube $\phi : G \times_H A \longrightarrow X$ about $Gx$ such that $\phi[e, A] = S$.*

(2) *$S$ is a slice at $x$.*

(3) *$GS$ is an open neighbourhood of $G(x)$ and there exists an equivariant retraction $r : GS \longrightarrow Gx$ such that $r^{-1}(x) = S$.*

**Proposition 2.6.** *(see* [2, II.4.4.]*) Let $X$ be a $G$-space and let $x \in S \subset X$. Then $S$ is a slice if and only if:*

1- *$S$ is closed in $G(S)$.*

2- *$G(S)$ is an open neighbourhood of $Gx$*

3- *$G_x(S) = S$.*

4- *If $(gS) \cap S \neq \emptyset$ then $g \in G_x$.*

The slice theorem asserts that, for certain conditions on the topological space and the group action, there exists a slice for every $x \in X$. The theorem was first proved by Gleason in the case of group actions is free. The proof of the general statement has contribution from Mongomery, Zippin, Koszul, Yang, Mostow and Palais.

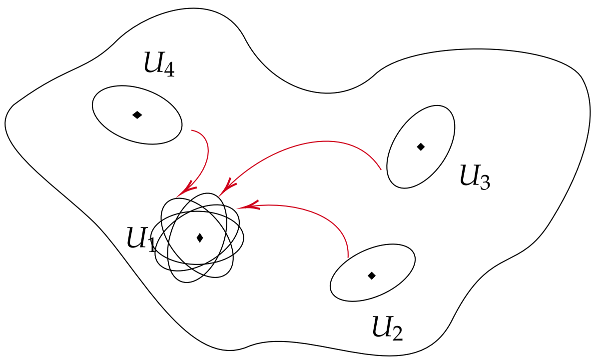
**Theorem 2.7.** *Let $G$ be a compact Lie group acting on a completely regular topological space $X$. Then there exists a slice for every $x \in X$.*

We start by proving the case where $G$ is a finite group. In this case, we only need $X$ to be Hausdorff:

**Theorem 2.8.** *Let $G$ be a finite group acting on a Hausdorff topological space $X$. Then there exists a slice for each $x \in X$.*

*Proof.* Let $x \in X$, then the orbit of $x$ is of the form $Gx = \{x_1, ..., x_r\}$ for some $r \in \mathbb{N}$ which divides the order of $G$. We assume that $x_1 = x$. Because $X$ is Hausdorff, there exists a collection of open neighbourhood $\{U_i\}_{i=1,...,r}$ such that $x_i \in U_i$ for all $i$ and $U_i \cap U_j \neq \emptyset$ if and only if $i = j$.

Now we consider the set $F = G \setminus G_x$. For each $g \in F$, we define $U_g = g^{-1}U_j \cap U_1$, where $j$ is such that $gx = x_j$. Note that $g(U_g) \subset U_j$. Therefore, we define the open neighbourhood of $x$, $V = \bigcap_{g \in F} U_g \subset U_1$ which fulfils that $g(V) \cap V = \emptyset$ for all $g \in F$.

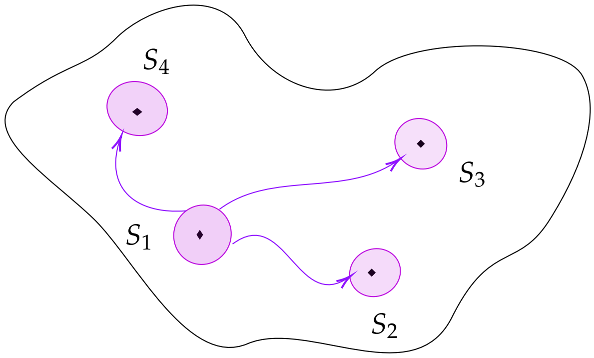



Finally, we define $S = \bigcap_{g \in G_x} g(V)$. The set $S$ is a $G_x$-invariant neighbourhood of $x$. Moreover, it fulfils that if $g(S) \cap g'(S) \neq \emptyset$ the $gG_x = g'G_x$. This is a consequence of the fact that $g(S)$ and $g'(S)$ are inside the same $U_j$, and therefore $gx = g'x$ and in consequence $g'g^{-1} \in G_x$.

Therefore, we define a map $\psi : G \times_{G_x} S \longrightarrow GS \subset X$ such that $\psi[g, s] = gs$. Note that $\psi$ fits into the commutative diagram

$$\begin{array}{ccc} G \times S & \xrightarrow{\Phi} & GS \\ \downarrow \pi & \nearrow \psi & \\ G \times_{G_x} S & & \end{array}$$

The map $\psi$ is injective since if $\psi[g, s] = \psi[g', s']$ then $g's' = gs$, which implies that $g^{-1}g's' = s$. By the choice of $S$, we have that $g^{-1}g' \in G_x$ and therefore $[g, s] = [g', s']$. The map is clearly surjective since $\Phi$ is surjective. The map is clearly continuos and it is also closed, since $\Phi$ is closed. Hence, we can conclude that $\psi$ is a homeomorphism.




If we define the action of $G$ on $G \times_{G_x} S$ given by $g'[g,x] = [g'g,x]$ then $\phi$ is $G$ equivariant. Thus $\psi$ is the desired tube and $S$ is a slice. □

The smooth version of the the slice theorem is one of the more important cases, and it is one of the main tools to proof the continuous version. We first proof the slice theorem when the action is free.

**Theorem 2.9.** *Let $G$ be a compact Lie group acting freely and smoothly on a smooth manifold $M$. Then there exists a slice for every $x \in M$.*

**Lemma 2.10.** *Let $G$ be a compact Lie group acting smoothly on a smooth manifold $M$ and let $x \in M$ such that $G_x = \{e\}$. Then $\alpha_x : G \longrightarrow M$ is an embedding and $Gx$ is a submanifold of $M$.*

*Proof.* We already know that $\alpha_x$ is a homeomorphism. Thus, we only need to proof that it is an immersion. Because $\alpha_x$ is $G$-equivariant, we have that $d\alpha_{|g} \cdot dg_{|e} = d\phi_{g|x} d\alpha_e$ for every $g \in G$, therefore we only need to prove that the map $d\alpha_e : T_eG = \mathfrak{g} \longrightarrow T_xM$ is injective.

In general, if $G_x$ is the isotropy subgroup at $x$ and $\mathfrak{g}_x$ be the Lie subalgebra of $G_x$, then we want to see that $\operatorname{Ker} d\alpha_e = \mathfrak{g}_x$. In order to proof this equality, we will use the exponential map $\exp : \mathfrak{g} \longrightarrow G$.

Recall that exp is characterized by the fact that for every $v \in \mathfrak{g}$, the curve $\gamma : \mathbb{R} \longrightarrow G$ such that $\gamma(t) = \exp(tv)$ is a group morphism and $\frac{d}{dt}\exp(tv)_{t=0} = v$. Moreover, if $H$ is a closed subgroup of $G$ than $\mathfrak{h} = \{v \in \mathfrak{g} : \exp(tv) \in H \text{ for all } t \in \mathbb{R}\}$.

For $v \in \mathfrak{g}$, we have that $d\,\mathrm{ev}_{x|e}(v) = \frac{d}{dt}(\exp(tv)x)_{|t=0}$. If $v \in \mathfrak{g}_x$, then $\exp(tv) \in G_x$ and the curve $t \mapsto \exp(tv)x$ is constant. In consequence $v \in \operatorname{Ker} d\,\mathrm{ev}_{|e}$. Conversely, assume that $v \in \operatorname{Ker} d\,\mathrm{ev}_{|e}$ and consider the curve $x \mapsto \exp(tv)x$, then for all $s \in \mathbb{R}$ we have that $\frac{d}{dt}(\exp(tv)x)_{|t=s} = \exp(sv)_* \frac{d}{dt}(\exp(tv)x)_{|t=0} = 0$. This means that $x \mapsto \exp(tv)x$ is a constant curve, which means that $\exp(tv) \in G_x$ and $v \in \mathfrak{g}_x$.

Finally, since $G_x = \{e\}$ (which means that $\mathrm{ev}_x = \alpha_x$), we have that $\mathfrak{g}_x = \{0\}$ and $\alpha_x$ is an immersion. □

Therefore, the smooth and free version of the slice theorem tells us that we have a tube $G \times A \longrightarrow M$ where $A$ is a smooth manifold, and moreover the tube is an embedding.

*Proof.* Firstly, note that the smooth action of $G$ on $M$ induces a smooth action of $G$ on the total space of tangent bundle $TM$, which fulfils that $g(y,v) = (gy, dg_{|y}(v))$ for all $y \in M$ and $v \in T_yM$ and $g \in G$. We want to find a $G$-invariant Riemannian metric on $M$. Given an arbitrary metric $\rho$, we define a new metric $\rho_G$ such that $\rho_G(v,w)_y = \int_G \rho(dg_{|y}(v), dg_{|y}(w))_{gy} d\mu(g)$ for all $v, w \in T_xM$, where we use the Haar measure. This metric is clearly $G$-invariant and therefore $G$ acts by isometries on $M$.

Recall that the exponential map $\exp : U \subset TM \longrightarrow M$ goes from an open neighbourhood $U$ of $\{(y,0) : y \in M\} \subset TM$ to $M$ such that $\exp(y,v) = \gamma_{(y,v)}(1)$, where $\gamma_{(y,v)} : [0,1] = I \longrightarrow M$ is the unique geodesic such that $\gamma_{(y,v)}(0) = x$ and $\gamma'_{(y,v)}(0) = v$. With the chosen $G$-invariant metric, the exponential map is equivariant. We have that $\exp(g(y,v)) = \gamma_{(gy,dg_{|y}(v))}(1)$. On the other hand, we have that the image of the geodesic of $\gamma_{(y,v)} : I \longrightarrow M$ by $g$ is a geodesic $g(\gamma_{(y,v)}) = \tilde{\gamma} : I \longrightarrow M$ such $\tilde{\gamma}(0) = gy$ and $\tilde{\gamma}'(0) = dg_{|y}(v)$, since $g$ is and isometry. By the uniqueness of geodesics, we have that $\tilde{\gamma} = \gamma_{(gy,dg_{|y}(v))}$ and in consequence $\exp(g(y,v)) = g\exp(y,v)$ for all $g \in G$ and $(y,v) \in U$.

Now we fix a point $x \in M$. Since $G$ is compact, $Gx$ is a compact submanifold of $M$ and we can decompose the tangent bundle over $Gx$ as $TM_{|Gx} = T(Gx) \oplus N$, where $N$ is the normal bundle of $Gx$ in $M$. It is well known that for $\epsilon$ small enough, the exponential exp maps $N(\epsilon) = \{v \in N : ||v|| < \epsilon\}$ diffeomorphically to $B(Gx,\epsilon) = \{y \in M : d(y, Gx) < \epsilon\}$, where the norm and the distance is induced by $\rho_G$. Note that we can identify $Gx$ with the point $(gx, 0) \in N(\epsilon)$.

If we look at the point $x$, we have a decomposition $T_xM = T_xGx \oplus N_x$. We consider the disk $V_\epsilon = \{v \in N_x : ||v|| < \epsilon\}$ and the map $\psi G \times V_\epsilon \longrightarrow M$ such that $\psi(g,v) = g\exp_x(v)$. We want to see that $\psi$ is a tube about $Gx$ and therefore $S = \exp_x(V_\epsilon)$ is a slice at $x$.

Since $G_x = \{e\}$, we have trivially that $G_xS = S$. Now we want to prove that if $gS \cap S \neq \emptyset$ then $g \in G_x$. Suppose that there exists $v, w \in V_\epsilon$ and $g \in G$ such that $g\exp(x,v) = \exp(x,w)$. Then $\exp((gx, dg_xv)) = \exp(x,w)$ and since exp is a diffeomorphism we obtain that $gx = x$, which means that $g \in G_x$.

Finally, we would like to prove that $\psi : G \times V_\epsilon \longrightarrow GS \subset M$ is a $G$-equivariant diffeomorphism. Let $r : N(\epsilon) \longrightarrow Gx$ be the retraction such that $r(gx,w) = (gx,0)$ and recall that $\alpha_x : G \longrightarrow Gx$ is a diffeomorphism. Then we define the map $K : G \times V_\epsilon \longrightarrow N(\epsilon)$ such that $K(g,v) = (gx, dg_xv)$. This map is continuous and smooth and has a smooth inverse $K^{-1}(y,v) = (\alpha^{-1}(r(y)), d(\alpha^{-1} \circ r)^{-1}_{|y}(v)))$. This implies that $\exp \circ K = \psi$ is a diffeomorphism. □

As a corollary, we obtain:

**Corollary 2.11.** *Let $M$ be a smooth manifold and let $G$ be a compact Lie group acting smoothly and freely on $M$. Then $M/G$ is a smooth manifold of dimension* $\dim M - \dim G$ *and $\pi : M \longrightarrow M/G$ is a smooth principal bundle.*

*Proof.* Given $Gx \in M/G$, we consider the tube $\psi : G \times V_\epsilon \longrightarrow GS$ about $Gx$ and slices as the proof of the theorem. Note that $\pi$ maps a slice $S$ homeomorphically to an open set of $M/G$. Then, the charts that give a smooth structure on $M/G$ are of the form $(\pi(S), \phi)$ where $\phi = (\pi \circ \exp)^{-1}$ : $\pi(S) \longrightarrow V_S \subset \mathbb{R}^n$, where $n = \dim M - \dim G$.

The local trivializations charts of the principal $G$-bundle are $(\pi(S), \chi)$ are precisely $\psi \circ (Id_G, \phi)$ : $G \times \pi(S) \longrightarrow GS$. □

Let $G$ be a compact Lie group and $H$ a closed subgroup, then the coset space $G/H$ is obtained as a quotient space by the free action of $H$ on $G$ by right translations.

**Proposition 2.12.** *Let $G$ be a (compact) Lie group and $H$ a closed subgroup, then $G/H$ is a smooth manifold and $\pi : G \longrightarrow G/H$ is a principal bundle.*

*Moreover, given any smooth $H$-space $F$ then the associated bundle $\pi : G \times_H F \longrightarrow G/H$ is smooth and $G \times_H F$ is a smooth manifold.*

In particular, we have a smooth local cross section $\sigma : U \longrightarrow G$, where $U$ is an open subset of $G/H$ containing $eH$ (recall that a local cross section is a map $\sigma : U \longrightarrow G$ such that $\pi \circ \sigma = Id_U$).

**Lemma 2.13.** *If $G$ is a compact Lie group acting on a smooth manifold $M$, then the map $\alpha_x : G/G_x \longrightarrow M$ is an embedding (hence $Gx$ is a submanifold of $M$).*

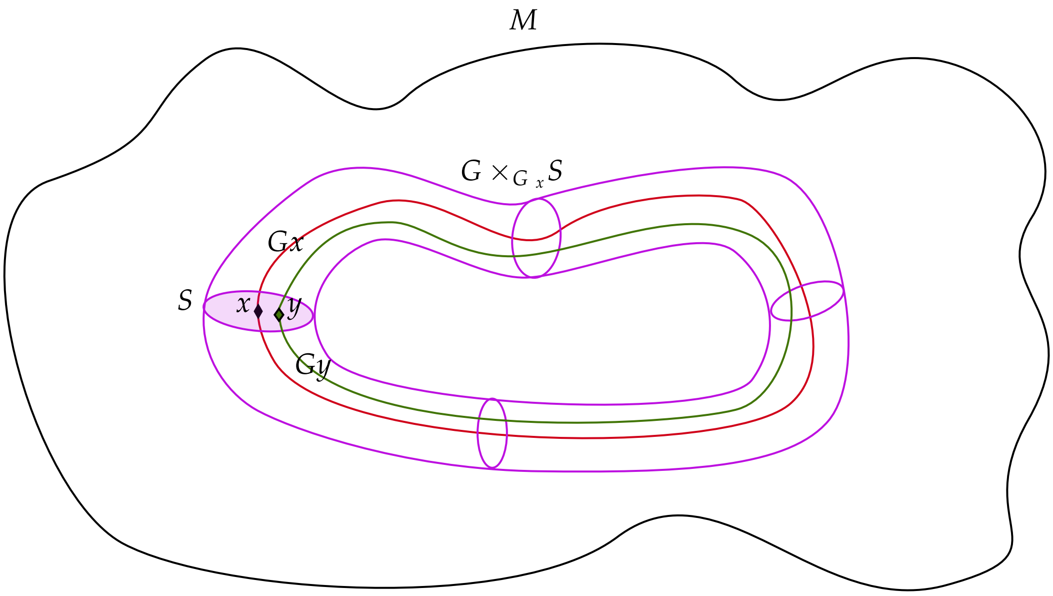
This is a consequence of the fact that $\operatorname{Ker} d\,\mathrm{ev}_{x|e} = \mathfrak{g}_x$.

**Theorem 2.14.** *Let $G$ be a compact Lie group acting smoothly on a smooth manifold $M$. Then there exists a slice for every $x \in M$.*

*Proof.* As before, we fix a $G$-invariant metric such that exp is $G$-equivariant. Now we fix a point $x \in M$. Since $G$ is compact, $Gx$ is a compact submanifold of $M$ and we can decompose the tangent bundle over $Gx$ as $TM_{|Gx} = T(Gx) \oplus N$, where $N$ is the normal bundle of $Gx$ in $M$. It is well known that for $\epsilon$ small enough, the exponential exp maps $N(\epsilon) = \{v \in N : ||v|| < \epsilon\}$ diffeomorphically to $B(Gx, \epsilon) = \{y \in M : d(y, Gx) < \epsilon\}$, where the norm and the distance is induced by $\rho_G$.

If we look at the point $x$, we have a decomposition $T_xM = T_xGx \oplus N_x$. Note that $G_x$ induces a linear (in fact orthogonal) action of $G_x$ on $T_xM$, given by the differential maps $dg_{|x} : T_xM \longrightarrow T_xM$. The action can be restricted to $N_x$, since $N_x$ is an invariant subspace. The disk $V_\epsilon = \{v \in N_x : ||v|| < \epsilon\}$ is $G_x$-invariant. Indeed, given $g \in G_x$ and $v \in V_\epsilon$, we have that $\exp(x, dg_{|x}v) = \exp(g(x, v)) = g\exp(x, v)$. Since $G$ acts by isometries we have that $g\exp(x, v) \in B(Gx, \epsilon)$. Finally, we use the fact that $\exp : N(\epsilon) \longrightarrow B(Gx, \epsilon)$ is a diffeomorphism to conclude that $gv \in V_\epsilon$.

Let $S = \exp(V_\epsilon)$. We want to see that $S$ is a slice at $x$. We have already proved that $G_xS = S$. Now we want to prove that if $gS \cap S \neq \emptyset$ then $g \in G_x$. Suppose that there exists $v, w \in V_\epsilon$ and $g \in G$ such that $g\exp(x, v) = \exp(x, w)$. Then $\exp((gx, dg_xv)) = \exp(x, w)$ and since exp is a diffeomorphism we obtain that $gx = x$, which means that $g \in G_x$.




Finally, we would like to see $\psi : G \times_{G_x} V_\epsilon \longrightarrow GS \subset M$ such that $\psi([g,v]) = g\exp_x(v)$ is a $G$-equivariant diffeomorphism. Like in the free case, $\psi$ is a bijective smooth map. We want to construct the same diffeomorphism as in the free case, but in this case we can only do it locally We consider the principal bundle $\pi : G \longrightarrow G/G_x$ with fiber $G_x$ and a local cross section $\sigma : U \longrightarrow G$, where $U$ is an open neighbourhood of $eG_x$. Therefore, we can define a diffeomorphism $K : U \times V_\epsilon \longrightarrow W$, where $W$ is an open set of $N_\epsilon$, such that $K(u,v) = (\sigma(u)x, d\sigma(u)_{|x}(v))$. Therefore, $\exp \circ K$ is a diffeomorphism onto an open set in $S$. Finally, since $\exp(\sigma(u)x, d\sigma(u)_{|x}(v)) = \sigma(u)\exp(x,v)$, we have that the map $(u,s) \longrightarrow \sigma(u)s$ is a diffeomorphism from $S \times U$ to an open set of $GS$. In particular, we have seen that $\psi$ is a local diffeomorphism for any point of the from $(e,v) \in G \times V_\epsilon$, but since $\psi$ is equivariant, we can conclude that $\psi$ is a diffeomorphism. □

With the a similar proof, we have the following generalization:

**Theorem 2.15.** *Let $G$ be a compact Lie group acting on a smooth manifold $M$ and let $A$ be a $G$-invariant submanifold. Then $A$ has a $G$-invariant normal neighbourhood.*

With the smooth version of the slice theorem we can also prove the continuous version of the theorem, by using some extra tools which we present now:

**Theorem 2.16.** *(see* [2, 0.5.2.]*) Let $G$ be a compact Lie group and $H$ a closed subgroup. Then there exists a representation $\rho : G \longrightarrow \mathrm{O}(m)$, for some $m$, and a point $v \in \mathbb{R}^m$ such that $G_v = H$.*

**Theorem 2.17.** *(Tietze-Gleason lemma,* [2, I.2.3.]*) Let $G$ be a compact group acting on a completely regular space $X$ and let $A$ be a compact invariant subspace. Let $\rho : G \longrightarrow \mathrm{GL}(n,\mathbb{R})$ be a representation of $G$ and let $f : A \longrightarrow \mathbb{R}^n$ be $\rho$-equivariant ($f(ga) = \rho(g)f(a)$). Then there exists an equivariant extension $\tilde{f} : X \longrightarrow \mathbb{R}^n$ of $f$.*

*Proof.* Because of the Tietze extension theorem, there exists a continuous map $f' : X \longrightarrow \mathbb{R}^n$ extending $f$. We define $\tilde{f}$ by averaging $f'$ with a normalized Haar integral on $G$. We define

$\tilde{f}(x) = \int \rho(g^{-1})f'(gx)d\mu(g)$. For all $h \in G$, we have that $\tilde{f}(hx) = \int \rho(h(gh)^{-1})f'(ghx)d\mu(g) = \int \rho(h)\rho((gh)^{-1})f'(ghx)d\mu(g) = \rho(h)\int \rho(k^{-1})f'(kx)d\mu(k) = \rho(h)\tilde{f}(x)$, and hence $\tilde{f}$ is equivariant. □

Now we can prove the continuous version of the slice theorem. Let $x \in X$ and suppose that the orbit $Gx$ and isotropy $H$. By theorem 2.16, there exists and orthogonal action of $G$ on some $\mathbb{R}^m$ and $v \in \mathbb{R}^m$ such that $Gv \cong G/H$. Since the action of $G$ on $\mathbb{R}^m$ is linear (and therefore smooth), there exists a slice $V$ for $v$ and a tube $G \times_H S \longrightarrow \mathbb{R}^m$. Now we define the continuous map $f : Gx \longrightarrow \mathbb{R}^n$ such that $f(gx) = gv$.Since $Gx$ is compact and invariant, we can use theorem 2.17 to find an equivariant extension $\tilde{f} : X \longrightarrow \mathbb{R}^m$. We consider $T = \tilde{f}^{-1}(GV)$ is a tube about $Gx$ and $S = \tilde{f}^{-1}(V)$ is the desired slice, since $GT = T$ is an open neighbourhood of $T$ we have an equivariant retraction $T \xrightarrow{\tilde{f}} GS \to G(v) \xrightarrow{f^{-1}} G(x)$.

**Remark 2.18.** *We have seen that if $X$ is a completely regular topological space and $G$ is a compact Lie group acting on $X$, then there exists a tube about each orbit. Moreover, this tube can be chosen to be of the form $G \times_{G_x} V \longrightarrow M$, where $V$ is a vector space and $H$ acts linearly on $V$. This type of action are called locally smooth actions.*

The slice theorem has far-reaching consequences. As a first corollary, we have the next structure results:

**Theorem 2.19.** *Suppose that $X$ is a completely regular space with an action of a compact Lie group $G$ and that all orbits have type $G/H$. Then the orbit map $\pi : X \longrightarrow X/G$ is a fiber bundle with fiber $G/H$ and structural group $N(H)/H$ acting by right translations on $G/H$.*

As another example, we proof the fact that if $G$ is a compact Lie group acting on $M$, then there exists a maximal isotropy type $(H)$ and $M_{(H)}$ is open and dense on $M$. We will denote by $x^*$ the orbit of $x$ in $M^* = M/G$ to make the notation more concise.

We prove the fact by induction on the dimension of $M$. We start by proving it a local version for a tube. Let $G \times_H V$ be a linear tube about $Gx$. By construction, we can assume that $H$ acts orthogonally on the unit sphere $S \subset V$. Now $\dim S < \dim M$ and the action of $H$ on $S$ is locally smooth, thus by the induction hypothesis there exists $K \leq H$ such that $S_{(K)}$ is open and dense. By seeing $V \setminus \{0\}$ as an open cone $\mathbb{R}_+ \cdot S$, we have that $V_{(K)}$ is open and dense in $V$ and all other orbits have isotropy type bigger or equal than $(K)$. Since two conjugated subgroups of $H$ are also conjugate in $G$, we have that $(G \times_H V)_{(K)} = G \times_H (V_{(K)})$.

Therefore we have found an open neighbourhood $U^*_x$ of $x^*$ and a connected open dense subset $W^*_x$ of $U^*_x$ such that all orbits in $W^*_x$ have the same type and all orbits in $U^*_x \setminus W^*_x$ are of strictly smaller type.

Let $C_{(H)}$ denote the closure of $M^*_{(H)}$, then $x^* \in C_{(H)}$ if and only if the orbits in $W^*_x$ have type $G/H$.. In this case, we have that $U^*_x \subset C_{(H)}$ and $C_{(H)}$ is open and closed. Then, there exists $H$ (now fixed) such that $C_{(H)} = M^*$. Since $M^*_{(H)} \cap U^*_x = W^*_x$ we have that $M^*_{(H)}$ is open dense and connected in $M^*$. Also all other orbits are of type strictly less than $G/H$.

## 3. The toral rank conjecture

Since the theory of transformation groups want to study the symmetry of topological spaces and manifolds, we would like to find a way to quantify how symmetric these spaces are. The toral rank is one of such ways, however we start by defining the degree of symmetry.

**Definition 3.1.** *Let $X$ be a Hausdorff topological space, then the degree of symmetry of $X$ is*

$$N(X) = \max\{\dim(G) : \textit{compact Lie group } G \textit{ acting effectively on } X\}.$$

We have the next result if we restrict the problem to smooth manifolds:

**Proposition 3.2.** *Let $M$ be a connected smooth manifold of dimension $n$, then $N(M) \leq \frac{n(n+1)}{2}$.*

*Proof.* We proof it by induction on the dimension of $M^n$. When $n = 0$, we have that the trivial group acts effectively yon $M$, since $M$ is a point. Assume now that $G$ is a connected compact Lie group acting on $M$ and consider a principal orbit $G/H$, then $\dim G - \dim H = \dim(G/H) \leq \dim M = n$. In consequence, $\dim G \leq n + \dim H$. If we find a manifold of dimension less than $n$ with an effective action of $H$ we are done.

Note that without loss of generality, we may assume that $H$ is connected. The action of $G$ on $G/H$ induced by the action of $G$ on $M$ is effective, therefore $H$ acts effectively on $G/H$. Therefore, we have a principal orbit $H/K$ of the action $H$ on $G/H$. If $\dim H/K < \dim G/H \leq n$ we are done. If $\dim G/H = \dim H/K$ then $G/H = H/K$, which means that there exists a transitive action with a fixed point on $G/H$. This implies that $G/H$ is a point and $G$ is the trivial group. In this case, we trivially have that $\dim G \leq \frac{n(n+1)}{2}$. □

If we pick $M = S^n$ then $\mathrm{SO}(n+1)$, which has dimension $\frac{n(n+1)}{2}$, acts effectively on $S^n$, thus the bound cannot be improved in general. Conversely, if a compact Lie group of dimension $\frac{n(n+1)}{2}$ acts smoothly (and thus by isometries after fixing a Riemannian metric) on a smooth manifold of dimension $M$, then $M$ is diffeomorphic to $S^n$ or $\mathbb{R}P^n$ (this is a classical result by Frobenius and Birkhoff). Note that the it is important to distinguish between topological and smooth actions. We can define $N^s(M)$ to be the smooth degree of symmetry of a smooth manifold $M$. If $\Sigma^n$ is an exotic sphere of dimension $n \geq 40$, then Hsiang showed that $N^s(\Sigma^n) \leq \frac{1}{8}n^2 + 1$, however $\Sigma^n$ is homeomorphic to $S^n$, which implies that $N(\Sigma^n) = \frac{n(n+1)}{2}$.

It is important to note that $N(M_1 \times M_2) \neq N(M_1) + N(M_2)$, but we have for example a result by Ku, Mann, Sicks and Su that $N^s(M_1 \times M_2) \leq \frac{n_1(n_1+1)}{2} + \frac{n_2(n_2+1)}{2}$ and the equality only happens if we have a product of spheres or real projective spaces.

However, most of the manifolds have few symmetries, there are manifolds which have $N(M) = 0$. For example, any closed surface $S$ other than $S^2$, $T^2$, $\mathbb{R}P^2$ or the Klein bottle $K$ fulfils that $N(S) = 0$. Another important results by Atiyah and Hizerbruch says that a spin manifold $M$ of dimension $4n$ with $\hat{A}(M) \neq 0$ has $N^s(M) = 0$. Another statement due to Montgomery and Conner says that if $M$ is a $K(\pi, 1)$ with $\chi(M) \neq 0$ then $N(M) = 0$. We can say even more, there exists asymmetric manifolds, which means that they do not admit any effective group action of a compact group, even finite groups. For example Waldmüller constructed a flat manifold ($\mathbb{R}^n/\Gamma$ with $\Gamma \subset E(n)$) with $n = 141$ which is asymmetric (see [7, 10] and the references therein for these results).

Since every compact Lie group has a maximal torus subgroup, every space which admits an effective action of a compact Lie group also has an effective action of a torus. Thus we can define:

**Definition 3.3.** *Let $X$ be a Hausdorff topological space, then the abelian degree of symmetry of $X$ is*

$$N_a(X) = \max\{r : T^r \textit{ acts effectively on } X\}.$$

Like in the preceding case, we have the next result for topological manifolds:

**Proposition 3.4.** *Let $M$ be a topological manifold of dimension $n$, then $N_a(M) \leq n$ and $N_a(M) = n$ if and only if $M$ is homeomorphic to $T^n$.*

*Proof.* Assume that $M$ admits an effective action of $T^r$ for some $r$. Then for every non trivial subgroup $K$ of $T^r$, the set of fixed points $M^K$ is closed and has empty interior. Since manifolds are Baire spaces, we have that $M \setminus \bigcap_{\{e\}\neq K\leq T^r} M^k \neq \emptyset$. Therefore, there exists $x \in M$ with trivial isotropy group. Therefore the map $f : T^r \longrightarrow M$ such that $f(t) = tx$ is continuous and injective, which implies that $r \leq \dim M$. If $r = \dim M$, then the map $f(T)$ is open. Since $f(T)$ is also closed and $M$ is connected we have that $M = f(T)$. Since $f$ is continuous, injective and open, we have that $f$ is a homeomorphism between $M$ and $T^r$. □

Note that the proposition shows that if the abelian degree of symmetry is high enough, we can determine properties of the manifold (and even the manifold itself!). The toral rank conjecture asserts a similar statement. If the toral rank of a space $X$ is big, then the cohomology of $X$ will be also big. Firstly, we need to define the toral rank of $X$.

**Definition 3.5.** *Let $X$ be a Hausdorff topological space, then the toral rank of $X$ is*

$$\operatorname{rank}(X) = \max\{r : T^r \textit{ acts almost-freely on } X\}.$$

Note that $\operatorname{rank}(X) \leq N_a(X)$ and in general we will have an strict inequality. This can be illustrated with the next proposition (see [3, Theorem 7.8] for a proof):

**Proposition 3.6.** *Assume that $M$ is a compact manifold which admits an almost free action of a compact Lie group $G$, then $\chi(M) = 0$.*

In particular, let $M = S^2$ which has $\chi(S^2) = 2$, by the above proposition we have that $\operatorname{rank}(S^2) = 0$ but $N_a(S^2) = 1$ since $S^1$ acts effectively on $S^2$ by a rotation around an axis (and $N_a(S^2) \neq 2$ since $S^2$ is not homeomorphic to $S^2$). On the other hand, we have that $N_a(T^n) = \operatorname{rank}(T^n) = n$.

In general, if $M$ is a closed aspherical (or more in general admissible) manifold, we have that $\operatorname{rank}(M) = N_a(M) \leq \dim \mathcal{Z}(\pi_1(M))$. This is a consequence of the fact that effective toral actions on admissible manifolds are injective (which means that if $T^r$ acts effectively on $M$, the group morphism $\mathrm{ev}_\# : \pi_1(T^r, e) \longrightarrow \pi_1(M, x)$ is injective for all $x \in X$, where $\mathrm{ev}(a) = ax$ for all $a \in T^r$). Clearly all injective toral actions are almost-free, since the map $\mathrm{ev}_\#$ factors through $\pi_1(T^r) \longrightarrow \pi_1(T^r/T^r_x)$.

On the other, not all almost-free actions are injective. For example, $S^1$ acts freely on $S^3$ (which induces the Hopf fibration), but $\mathrm{ev}_\#$ is the trivial map since $S^3$ is simply connected. Generally, if

$M$ is a simply connected closed manifold of dimension $n \geq 3$, we have that $N_a(M) \leq n-2$ and $\text{rank}(M) < n-2$ for $n \geq 5$ (see [6]).

Let $\dim H^*(X, \mathbb{Q}) = \sum_{i \geq 0} \dim H^i(X, \mathbb{Q})$, then:

**Conjecture 3.7.** *(Toral rank conjecture) Let $X$ be a compact Hausdorff topological space, then* $\dim H^*(X, \mathbb{Q}) \geq 2^{\text{rank}(X)}$.

We refer to [3, 9] for an introduction to the conjecture.

The original statement of the conjecture was due to S. Halperin and asked if $\dim H^*(X, \mathbb{Q}) \geq 2^{\text{rank}(X)}$ for a simply connected *reasonable* space, defining reasonable as a technical condition in order to use rational homotopy theory and some results of topological transformation groups. Spaces like finite CW-complexes and compact manifolds fulfils the condition of being reasonable.

Note that the almost-free condition is essential, so we cannot replace $\text{rank}(X)$ with $N_a(X)$. Indeed, $\text{U}(n)$ acts effectively on $S^{2n-1} \subset \mathbb{C}^n$, which implies that $T^n = \text{U}(1)^n \subset \text{U}(n)$ also acts effectively on $S^{2n+1}$. On the other hand $\dim H^*(S^{2n-1}) = 2 < 2^n$ for $n > 1$.

The conjecture can be reformulated by saying that $\text{rank}(X) \leq \log_2(\dim H^*(X, \mathbb{Q}))$ for any compact Hausdorff topological space. Note that $2^{\text{rank}(X)} = \dim H^*(T^{\text{rank}(X)}, \mathbb{Q})$. In consequence, if the action of $T^r$ on $X$ is free and we have the principal bundle $\pi : X \longrightarrow X/T^r$, then the toral rank conjecture predicts that $\dim H^*(X) \geq \dim H^*(T^r)$. This is true for example when the principal bundle is trivial, since we would have that $\dim H^*(X) = 2^r \dim H^*(X/G) \geq 2^r$. However, the conjecture does not tells us that $\dim H^i(X) \geq \dim H^i(T^r)$ for all $i \geq 0$. Indeed, this is not true in general, as the case of $S^1$ acting freely on $S^3$ shows.

However, we have the following affirmative (and stronger) result in some particular cases. An action of $T^r$ on $X$ is called homologically injective if the map $\text{ev}_* : \mathbb{Z}^r \longrightarrow H^1(X, \mathbb{Z})$ is injective. Note that a homologically injective action is injective and thus almost free. Then:

**Theorem 3.8.** *Let $M$ be a closed manifold which admits a homologically injective action of $T^r$, then* $\dim H_i(T^r, \mathbb{Q}) \leq \dim H_i(M, \mathbb{Q})$ *for all $i$.*

Every effective action of a torus on a compact flat manifold $M$ is homologically injective (and there exists an effective action of $T^r$, where $r = \text{rank}\, \mathcal{Z}(\pi_1(M))) = \text{rank}(M)$). In conclusion, the conjecture is true for compact flat manifolds.

The toral rank conjecture can be generalized to finite group actions. Let $p$ be a prime number, then a $p$-torus of rank $r$ is the group $(\mathbb{Z}_p)^r$ (which it is also called elementary abelian $p$-group). We can define analogously the free $p$-rank of $X$ to be

$$\text{rank}_p(X) = \max\{r : (\mathbb{Z}_p)^r \text{ acts freely on } X\}.$$

**Conjecture 3.9.** *(Halperin-Carlsson conjecture) Let $X$ be a paracompact finite dimensional space, then* $\dim H^*(X, \mathbb{F}_p) \geq 2^{\text{rank}_p(X)}$ *and* $\dim H^*(X, \mathbb{Q}) \geq 2^{\text{rank}(X)}$.

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