SIMBA : Boundary orbits in holomorphic dynamical systems

1.- Discrete dynamical systems

X space, $f: X \to X$ transformation

For all $x \in X$, consider $\{f^n(x)\}_n$ ← Goal : understand the asymptotic behaviour (depends on X and f)

X	f	
measure space	measurable	→ Ergodic theory
metric / topological space	continuous	
$\mathbb{R}^n$ / manifold	smooth	
Riemann surface	holomorphic	→ holomorphic dynamics

Idea : we ask f to preserve the structure of X (to be compatible with the structure of X)

2.- 1D holomorphic dynamical systems

X (1-D) Riemann surface, $f: X \to X$ holomorphic

- up to (semi) conjugacy, we can take $X \in \{\mathbb{D}, \mathbb{C}, \hat{\mathbb{C}}\}$ (consequence of the uniformization thm for Riemann surfaces)
  (local) change of variables

Assume $\boxed{f: \mathbb{C} \to \mathbb{C} \text{ entire transcendental}}$

- case $X = \hat{\mathbb{C}}$ is similar (in fact easier due to compactness)
- later, $X = \mathbb{D}$

Totally invariant partition of $\mathbb{C}$:

- **Fatou set $F(f)$**
  - $z \in F(f) \Leftrightarrow \{f^n\}_n$ normal in a nbh of z
    $\underset{\text{Arzelà-Ascoli}}{\Leftrightarrow} \{f^n\}_n$ equicontinuous in a nbh of z

    Idea: if we take a point in the Fatou set, it will behave similarly as points nearby.
  - stable dynamics
  - open (by definition)
  - connected components → Fatou components

- **Julia set $J(f) := \mathbb{C} - F(f)$**
  - chaotic dynamics
  - closed

3.- Dynamics in the Fatou set

$\boxed{f: F(f) \longrightarrow F(f)}$

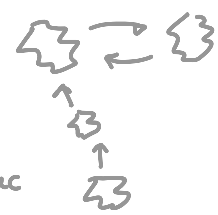
U Fatou component $\Rightarrow$ $f(U)$ Fatou component

↑

$F(f)$, $J(f)$ totally invariant

Open Mapping Principle

U FC. Three possibilities:



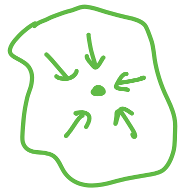
1. periodic $f^n(U) = U$ for some n
2. preperiodic U not periodic, but $f^n(U)$ periodic
3. wandering domain $f^n(U) \cap f^m(U) = \emptyset \quad \forall n \neq m$

etc.

- understanding the dynamics in the Fatou set = understanding the dynamics in periodic FC and wandering domains
- periodic FC can be assumed to be invariant
- wandering domains are difficult, and are not going to be considered in this talk

THM U invariant FC. 4 possibilities:

a) attracting basin

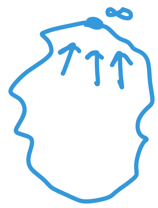


c) Siegel disk

$f|_U \simeq$ irrational rotation

b) parabolic basin

d) Baker domain



(comentar la diferència entre una parabolic basin i un BD)

<u>Idea of the proof</u> U invariant $\underset{\text{(Baker)}}{\Rightarrow}$ U simply connected

$$\begin{tikzcd} U \arrow[r, "f"] & U \\ \mathbb{D} \arrow[u, "\varphi"] \arrow[r, "g"] & \mathbb{D} \arrow[u, "\varphi"] \end{tikzcd}$$

– Riemann

$f|_U \simeq$ $\boxed{g: \mathbb{D} \longrightarrow \mathbb{D} \quad \text{holomorphic}}$ $\to$ iteration of holomorphic functions in $\mathbb{D}$

<u>Denjoy - Wolff theorem</u> either
- g is a rotation

or
- $\exists\, p \in \overline{\mathbb{D}}$ s.t. $g^n(z) \to p \quad \forall z \in \mathbb{D}$ //

↑ només assumim g holomorfa a $\mathbb{D}$ (no assumim cap extensió a la frontera)

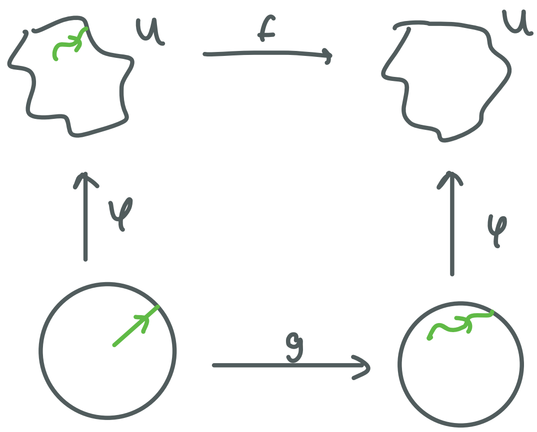
4.- Dynamics in the Julia set $\boxed{f: J(f) \to J(f)}$

CHAOS
- $\exists z \in J(f)$ st. $\overline{\{f^n(z)\}_n} = J(f)$ (its orbit is dense in $J(f)$)
- periodic points are dense
- $f|_{J(f)}$ is topologically transitive, i.e. cannot be decomposed in smaller dynamical systems

complicated: think of a dynamics and it will be somewhere realised (in contrast to FC on $F(f)$)

Forward partition of $J(f)$: $\boxed{\text{boundaries of FC}}$ $\to$ U invariant FC $\boxed{f: \partial U \to \partial U}$
- small
- use the conjugacy to $\mathbb{D}$ (as before)

TOOL Extend the conjugacy $f|_U \simeq g: \mathbb{D} \to \mathbb{D}$ to $f|_{\partial U} \simeq g: \partial\mathbb{D} \to \partial\mathbb{D}$



g, φ may not extend (continuously) to $\overline{\mathbb{D}}$, but

radial limits $g^*(e^{i\theta}), \varphi^*(e^{i\theta})$ exists a.e.

and $g^*: \partial\mathbb{D} \to \partial\mathbb{D}$ a.e.

measurable

$\varphi^*: \partial\mathbb{D} \to \partial U$ a.e. (measurable conjugacy $\to$ measurable change of variables)

$\to$ tools from ERGODIC THEORY

$\Rightarrow$ For a.e. point in ∂U, $\overline{\{f^n(z)\}_n} = \partial U$

(some BD are excluded) //

THM (Fagella, J.) $\mathcal{U}$ invariant FC, unbounded

- $\mathcal{U}$ Siegel disk $\Rightarrow$ no periodic points in $\partial\mathcal{U}$
- otherwise,

  (+ technical condition on SV)

  $\Rightarrow \left\{ \begin{array}{l} \text{periodic} \\ \text{escaping} \end{array} \right.$ points in $\partial\mathcal{U}$ are dense in $\partial\mathcal{U}$

TOOLS

- previous conjugacy

  +

  $\left\{ \begin{array}{l} g \text{ irrational rotation} \Rightarrow \text{no periodic points in } \partial\mathbb{D} \\ \text{otherwise, periodic points are dense in } \partial\mathbb{D} \end{array} \right.$ (Bargmann)

  +

  better estimates on φ^*

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