An introduction to Khovanov homology and annular links

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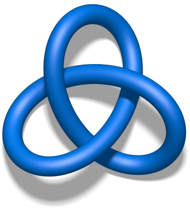

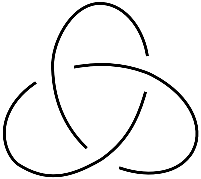

Definition

An annular knot K is a subset of the thickened annulus $\mathbb{A} \times I$ homeomorphic to a circumference S^1.
An annular link L is a finite disjoint union of annular knots $L = K_1 \cup K_2 \cup \cdots \cup K_n$.

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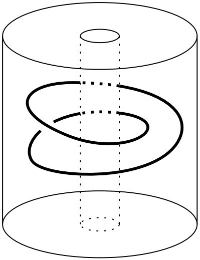

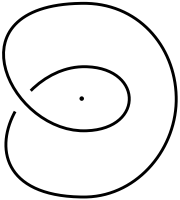

Definition

Two links L_1 y L_2 are equivalent if there exists an ambient isotopy $F : \mathbb{R}^3 \times [0, 1] \longrightarrow \mathbb{R}^3$ such that $F(L_1, 0) = L_1$ and $F(L_1, 1) = L_2$.

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Definition

A link invariant is a function from the set of (annular) links whose value depends only on the equivalence class of the link.

Theorem (Reidemeister, 1927)

Two diagrams D and D' represent equivalent links if and only if there is a finite sequence of Reidemeister moves that transform D into D'.

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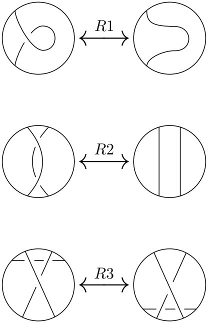

Theorem (Hoste and Przytycki, 1989)

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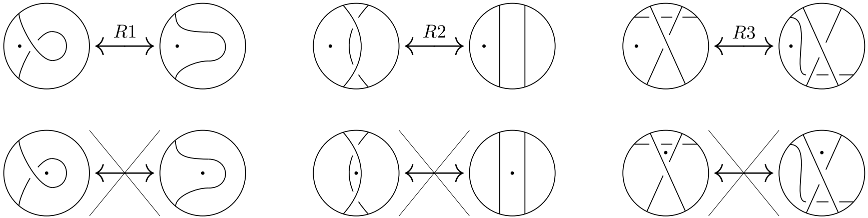
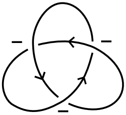
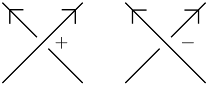


We can orientate a link choosing an orientation for each component.





Given an oriented diagram D, we call writhe of D to the number of positive crossing p minus the number of negative crossings n of D, $w(D) = p - n$.

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Definition

Let D be a non-oriented diagram. We define the Kauffman bracket of D as the polynomial $\langle D \rangle \in \mathbb{Z}[A^{\pm 1}]$ that satisfies the following axioms:

1. $\langle \bigcirc \rangle = 1$,
2. $\langle D \sqcup \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle$,
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Definition

Let D be an oriented diagram of a link L. We define the Jones polynomial of D as

$$V(D) = (-A)^{-3w(D)}\langle D\rangle.$$

It is a link invariant.

Definition

Let D be an annular diagram. We define the annular Kauffman bracket of D as the polynomial $\langle D\rangle_{\mathbb{A}} \in \mathbb{Z}[A^{\pm 1}, h]$ that satisfies the following axioms:

1. $\langle \cdot \bigcirc \rangle_{\mathbb{A}} = 1$,
2. $\langle \odot \rangle_{\mathbb{A}} = h$,
3. $\langle \times \rangle_{\mathbb{A}} = A\langle \asymp \rangle_{\mathbb{A}} + A^{-1}\langle)\,(\rangle_{\mathbb{A}}$,
4. $\langle \cdot \bigcirc \sqcup D\rangle_{\mathbb{A}} = (-A^2 - A^{-2})\langle \cdot D\rangle_{\mathbb{A}}$,
5. $\langle \odot \sqcup D\rangle_{\mathbb{A}} = (-A^2 - A^{-2})h\langle \cdot D\rangle_{\mathbb{A}}$.

Definition (Hoste and Przytycki)

Let D be an annular diagram of an oriented annular link L. We define the annular Jones polynomial of D as

$$V_{\mathbb{A}}(D) = (-A)^{-3w(D)} \langle D \rangle_{\mathbb{A}}.$$

It is an annular link invariant.

If L is an annular link contained in a 3-ball in $\mathbb{A} \times I$, then $V_{\mathbb{A}}(L) = V(L)$.

Definition

Let D be a diagram of a link L. We define the bracket polynomial D as the polynomial $\langle D \rangle \in \mathbb{Z}[q^{\pm 1}]$ that satisfies the following axioms:

1. $\langle\langle \bigcirc \rangle\rangle = 1$,
2. $\langle\langle D \sqcup \bigcirc \rangle\rangle = (q + q^{-1}) \langle\langle D \rangle\rangle$,
3. $\langle\langle \times \rangle\rangle = \langle\langle \asymp \rangle\rangle - q \, \langle\langle \,)\,(\, \rangle\rangle$.

Definition

Let D be a diagram of a link L, p and n the number of positive and negative crossings, respectively, of D. We define the normalized Jones polynomial (in the varaible q) of L as

$$J(L) = (-1)^n q^{p-2n} \langle\langle D \rangle\rangle.$$

Definition

Given a diagram D, a Kauffman state s of D is an assignment of a label 0 o 1 to each crossing of the diagram.

A 0-smoothing and a 1-smoothing is the result of swapping a crossing  for  and , respectively.

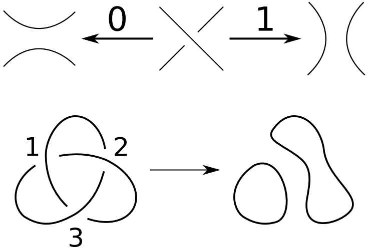
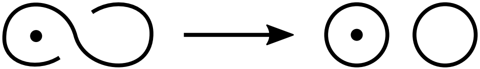


Figure: Kauffman state 010 of the trefoil knot.

A Kauffman state of an annular diagram has two types of circles:
Trivial circles, that bound a disk in the punctured plane.
Essential circles, that do not bound a disk in the punctured plane.



$$J(L) = (-1)^n q^{p-2n} \sum_s (-q)^r (q + q^{-1})^{k-1},$$

where r is the height or number of 1-smoothings of s,
k is the number of circles in s,
p is the number of positive crossings,
n is the number of negative crossings.

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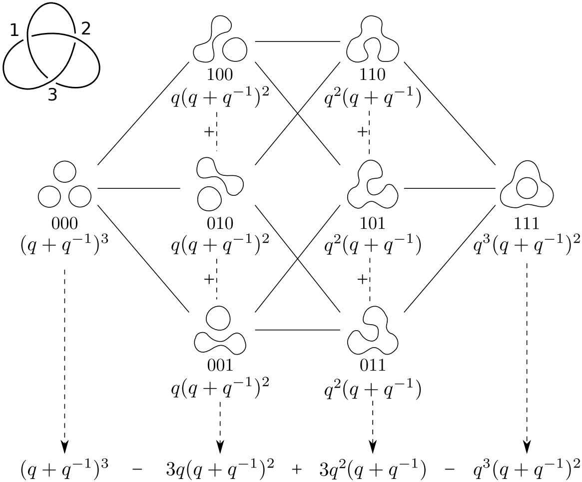
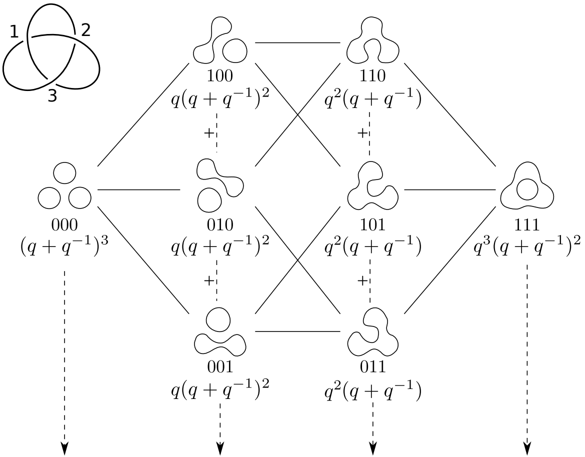
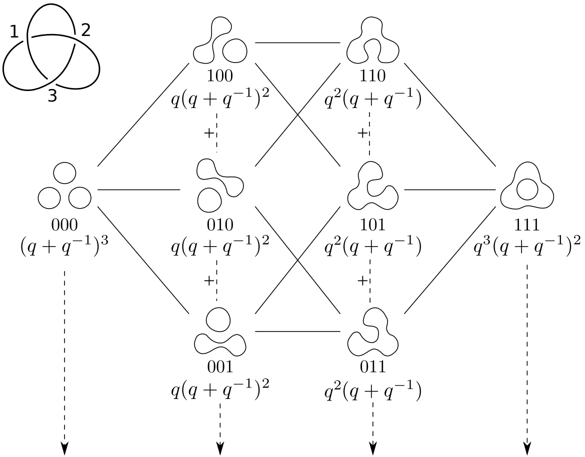




$$\langle\!\langle \quad \rangle\!\rangle = \frac{(q+q^{-1})^3 \; - \; 3q(q+q^{-1})^2 \; + \; 3q^2(q+q^{-1}) \; - \; q^3(q+q^{-1})^2}{q+q^{-1}} = q^{-2} - 1 - q^4$$



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$J($$) = (-1)^n q^{p-2n}$⟪⟫ $= (-1)^3 q^{-6}(q^{-2} - 1 - q^4) = -q^{-8} + q^{-6} + q^{-2}.$

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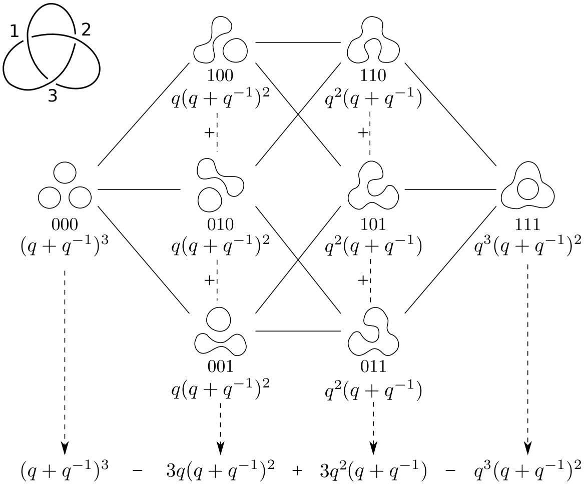
1
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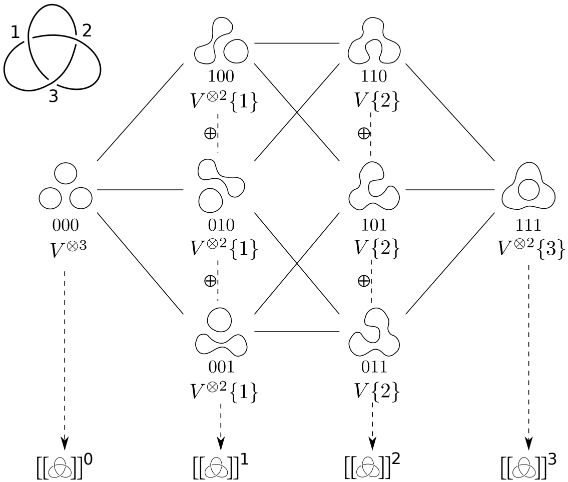
q(q + q^{-1})^2
110
q^2(q + q^{-1})
+
+
000
(q + q^{-1})^3
010
q(q + q^{-1})^2
101
q^2(q + q^{-1})
111
q^3(q + q^{-1})^2
+
+
001
q(q + q^{-1})^2
011
q^2(q + q^{-1})
(q + q^{-1})^3 − 3q(q + q^{-1})^2 + 3q^2(q + q^{-1}) − q^3(q + q^{-1})^2


1
2
3
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$V^{\otimes 2}\{1\}$
110
$V\{2\}$
000
$V^{\otimes 3}$
010
$V^{\otimes 2}\{1\}$
101
$V\{2\}$
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$V^{\otimes 2}\{1\}$
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$\oplus$
$[[\ ]]^0$
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$[[\ ]]^2$
$[[\ ]]^3$

$$
\begin{array}{rccc}
 & \bigcirc\bigcirc & \longmapsto & \bigcirc \\
m\colon & V \otimes V & \longrightarrow & V \\
 & v_+ \otimes v_+ & \longmapsto & v_+, \\
 & v_+ \otimes v_- & \longmapsto & v_-, \\
 & v_- \otimes v_+ & \longmapsto & v_-, \\
 & v_- \otimes v_- & \longmapsto & 0.
\end{array}
$$

$$
\begin{array}{rccl}
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$$

$$
\begin{array}{rccl}
 & \bigcirc & \longmapsto & \bigcirc\bigcirc \\
\Delta\colon & V & \longrightarrow & V \otimes V \\
 & v_+ & \longmapsto & v_+ \otimes v_- + v_- \otimes v_+, \\
 & v_- & \longmapsto & v_- \otimes v_-.
\end{array}
$$

$$\begin{array}{rcccl}
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 & v_- \otimes v_- & \longmapsto & 0. &
\end{array}$$

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 & \bigcirc & \longmapsto & \bigcirc\bigcirc \\
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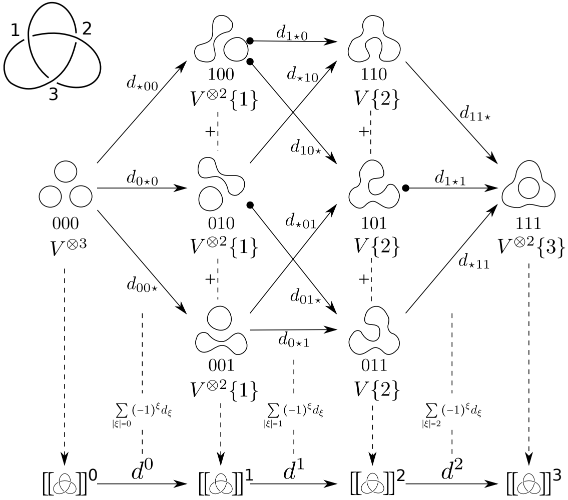
 & v_- & \longmapsto & v_- \otimes v_-\,.
\end{array}$$

$$d^r = \sum_{|\xi|=r} (-1)^{\xi} d_{\xi}.$$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$d_{\star 00}$
$d_{0\star 0}$
$d_{00\star}$
$d_{1\star 0}$
$d_{\star 10}$
$d_{10\star}$
$d_{\star 01}$
$d_{01\star}$
$d_{0\star 1}$
$d_{11\star}$
$d_{1\star 1}$
$d_{\star 11}$
+
$\sum_{|\xi|=0}(-1)^{\xi}d_{\xi}$
$\sum_{|\xi|=1}(-1)^{\xi}d_{\xi}$
$\sum_{|\xi|=2}(-1)^{\xi}d_{\xi}$
d^0
d^1
d^2

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The graded Poincaré polynomial of $\mathcal{C}(D)$,

$$\sum_r t^r \dim \mathcal{H}^r(D)$$

is a link invariant that recovers the Jones polynomial:

$$\tilde{J}(L) = \sum_r (-1)^r \dim \mathcal{H}^r(D).$$

$j \backslash i$	-3	-2	-1	0
-1				$\mathbb{Z}$
-3				$\mathbb{Z}$
-5		$\mathbb{Z}$		
-7		$\mathbb{Z}_2$		
-9	$\mathbb{Z}$			

$\sum_r t^r \dim \mathcal{H}^r($  $) = t^{-3}q^{-9} + t^{-2}q^{-5} + q^{-3} + q^{-1}.$

$J($  $) = \dfrac{-q^{-9} + q^{-5} + q^{-3} + q^{-1}}{q + q^{-1}} = -q^{-8} + q^{-6} + q^{-2}.$

Khovanov homology is a sharper invariant than Jones polynomial:

$$J(5_1) = J(10_{132}) = -q^{-7} + q^{-6} - q^{-5} + q^{-4} + q^{-2}.$$

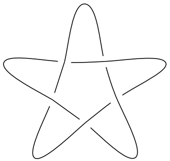

Figure: Diagram of 5_1.

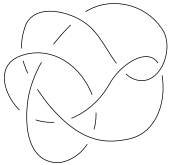

Figure: Diagram of 10_{132}.

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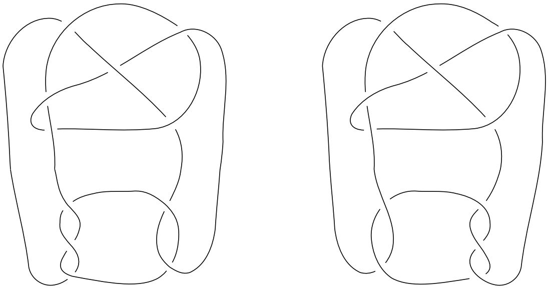
$\mathcal{H}^{i,j}(5_1)$

j \ i	−5	−4	−3	−2	−1	0
−3						$\mathbb{Z}$
−5						$\mathbb{Z}$
−7				$\mathbb{Z}$		
−9						
−11		$\mathbb{Z}$	$\mathbb{Z}$			
−13						
−15	$\mathbb{Z}$					

$\mathcal{H}^{i,j}(10_{132})$

j \ i	−7	−6	−5	−4	−3	−2	−1	0
−1							$\mathbb{Z}$	$\mathbb{Z}$
−3								$\mathbb{Z}$
−5					$\mathbb{Z}$	$\mathbb{Z}^2$		
−7				$\mathbb{Z}$				
−9				$\mathbb{Z}$	$\mathbb{Z}$			
−11		$\mathbb{Z}$	$\mathbb{Z}$					
−13								
−15	$\mathbb{Z}$							

Khovanov homology is not a complete invariant: it doesn't distinguish Kinoshita-Terasaka's knot and Conway's knot.



Theorem (Kronheimer and Mrowka, 2010)

Khovanov homology detects the trivial knot.

Annular Khovanov homology

We construct the resolution cube.

$\bigcirc \to V = \langle v_+, v_- \rangle.$ $\odot \to W = \langle w_+, w_- \rangle.$

$\deg_q(v_+) = 1, \deg_q(v_-) = -1,$ $\deg_q(w_+) = 0 = \deg_q(w_-),$

$\deg_h(v_+) = 0 = \deg_h(v_-).$ $\deg_h(w_+) = 1, \deg_h(w_-) = -1.$

$$V_s(D) = V^{\otimes a} \otimes W^{\otimes b}\{r\}, \text{ where } a = \#\bigcirc, b = \#\odot.$$

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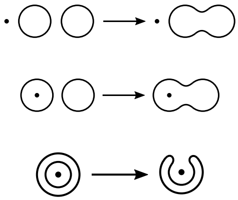
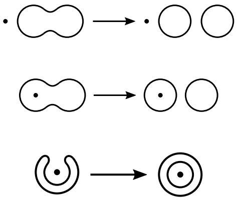
$$\dots \longrightarrow [[D]]^r_{\mathbb{A}} \xrightarrow{d^r} [[D]]^{r+1}_{\mathbb{A}} \xrightarrow{d^{r+1}} \dots$$

$$[[D]]^r_{\mathbb{A}} = \bigoplus_{s:|s|=r} V_s(D) = \bigoplus_{s:|s|=r} V^{\otimes a} \otimes W^{\otimes b}\{r\}.$$

Annular Khovanov complex: $\mathcal{C}_{\mathbb{A}}(D) = [[D]]_{\mathbb{A}}[-n]\{p-2n\}$.

We define maps m and Δ.





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Let D be an annular diagram of an oriented link L. We define the annular Khovanov homology of D, $\mathcal{H}_{\mathbb{A}}(D)$, as the set of homology groups of the annular Khovanov complex $\mathcal{C}_{\mathbb{A}}(D)$.

If L is an annular link contained in a 3-ball in $\mathbb{A} \times I$, then $\mathcal{C}_{\mathbb{A}}(D) = \mathcal{C}(D)$, hence $\mathcal{H}_{\mathbb{A}}(L) = \mathcal{H}(L)$.

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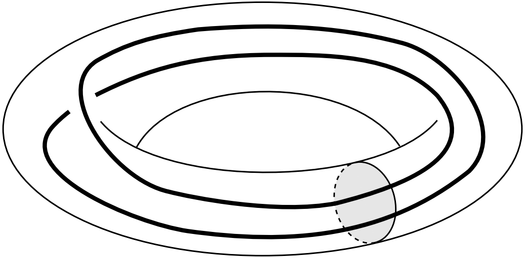
Theorem (Asaeda, Przytycki and Sikora, 2004)

The annular Khovanov homology $\mathcal{H}^r_{\mathbb{A}}(D)$ is an annular link invariant.

Annular Khovanov homology categorifies the annular Jones polynomial.

Definition

Given a link L in the solid torus $\mathbb{T}$, we define the wrapping number of L, $wrap(L)$, as the minimal intersection of L with a meridional disk of $\mathbb{T}$.



Conjecture (Wrapping conjecture. Hoste and Przytycki, 1995)

Let L be an annular link. Then, the maximum annular degree of $V_{\mathbb{A}}(L)$ coincides with the wrapping numberof L. That is,

$$\max \deg_h V_{\mathbb{A}}(L) = wrap(L).$$

Conjecture (Wrapping conjecture II)

Let L be an annular link. Then,

$$\max\{k | \mathcal{H}^{**k}(L) \text{ not trivial}\} = wrap(L).$$

Contents

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Thank you!